Eur. J. Mech. B/Fluids, **17**, n° 3, 277-306, 1998

Analytical study of transonic flows of a gas condensing onto its plane condensed phase on the basis of kinetic theory

Yoshio SONE [a*], François GOLSE [b], Taku OHWADA [a] and Toshiyuki DOI [c]

ABSTRACT. – A uniform flow of a gas condensing onto its plane condensed phase (commonly known as the half-space problem of condensation) is considered. The problem is studied analytically on the basis of the Boltzmann equation when the flow is in a transonic region. The paper clarifies the analytical structure of the solution, especially the mechanism by which the range of the parameters (the flow speed, pressure, and temperature of the uniform flow blowing from infinity) where a steady solution exists changes abruptly (from a surface to a domain in the parameter space) when the flow speed passes the sonic speed, the correspondence of a family of supersonic solutions to a subsonic solution, etc. The solutions constructed analytically are compared with new numerical solutions near the sonic point. © Elsevier, Paris

Keywords: Kinetic theory, condensation, rarefied gas.

1. Introduction

We consider a uniform flow of a gas condensing onto its plane condensed phase (commonly known as the half-space problem of condensation) on the basis of kinetic theory. This is important not only as a fundamental problem in rarefied gas dynamics but also as the problem giving the boundary condition for the basic equation in the continuum gas dynamics (Euler's set of equations) on the interface between a gas and its condensed phase (Aoki and Sone, 1991). The problem has been studied, analytically for weak condensation, by Sone (1978) or, numerically or approximately for the general case, by Sone *et al.* (1986), Aoki *et al.* (1990, 1991), Kogan and Abramov (1991), Kryukov (1991), Sone *et al.* (1992), and Aoki and Doi (1994). The range of the parameters (the flow speed, pressure, and temperature of the uniform flow) where a steady solution exists changes drastically when the flow speed passes the sonic speed: for a subsonic flow, the solution exists on a surface of the (three dimensional) parameter space, but for supersonic flow it exists in a domain in the space. This interesting feature urged mathematicians to tackle with the existence and uniqueness study of the problem (Cercignani, 1986; Coron *et al.*, 1988; Ghaoui and Golse, 1997). Partial success was obtained for a Boltzmann equation linearized around the uniform flow, but this does not explain the typical shape of the existence range in the supersonic region. (This failure comes from the fact that the speed of wave propagation in the linear system is independent of the amplitude of the wave.) In the present paper, we consider the problem in a transonic region and try to clarify the structure of the solution analytically: the mechanism of the abrupt change at the sonic point of the range of the parameters where a steady solution exists, the structure of the supersonic solution, the correspondence of a family of supersonic solutions to a subsonic solution, etc. We will also carry out detailed numerical computations of the flow very close to the sonic points, which supplement our previous works, and compare the analytical and numerical solutions.

[a] Department of Aeronautics and Astronautics, Graduate School of Engineering, Kyoto University, Kyoto 606-8501, Japan
[b] Ecole Normale Supérieure, DMI, 45 rue d'Ulm, 75005 Paris, France
[c] Department of Applied Mathematics and Physics, Tottori University, Tottori 680-8552, Japan
* Correspondence and reprints

0997-7546/98/03/© Elsevier, Paris

2. Problem and basic equation

We first describe the half-space problem of condensation precisely. Consider a semi-infinite expanse of a gas ($X_1 > 0$) bounded by its plane condensed phase (at $X_1 = 0$) with a uniform and constant surface temperature T_w, where X_i is the space rectangular coordinate system. The saturation gas pressure at temperature T_w is denoted by p_w. (The saturated gas pressure p_w is related to the temperature T_w by the Clausius-Clapeyron relation. In the present work, this relation is never used. Thus, p_w and T_w can be chosen freely in the results or in the discussions. The relation may be introduced when necessary.) A uniform flow of the gas [with flow velocity $(v_\infty, 0, 0)$ $(v_\infty < 0)$, pressure p_∞, and temperature T_∞] is blowing from infinity ($X_1 = \infty$). The half-space problem is to clarify the steady behaviour of the gas of the system described above. The steady state may not necessarily be possible for an arbitrary set of the parameters v_∞, p_∞, T_∞, p_w, and T_w. It is one of the main interests in the half-space problem to find the relations among the parameters that allow a steady solution. The problem is studied under the following assumptions: i) the behaviour of the gas is described by the Boltzmann equation; ii) the gas molecules leaving the condensed phase constitute the corresponding part of the stationary Maxwellian distribution pertaining the saturated gas at the surface temperature of the condensed phase. (The condition is called the conventional boundary condition of evaporation and condensation or the complete condensation condition.) It is shown, for example, by Aoki and Sone (1991), Sone *et al.* (1992), and Sone and Aoki (1994) that the solution for a more general boundary condition can be easily constructed from the solution under the above conventional boundary condition.

The one-dimensional Boltzmann equation for a steady state is written in the following nondimensional form:

$$\zeta_1 \frac{\partial \hat{\Phi}}{\partial x} = \hat{J}(\hat{\Phi}, \hat{\Phi}), \tag{2.1}$$

where $(2RT_w)^{1/2}\zeta_i$ (R: the specific gas constant or the Boltzmann constant k divided by the mass m of a molecule) is the molecular velocity, $p_w(RT_w)^{-5/2}\hat{\Phi}/2^{3/2}$ is the velocity distribution function of the gas molecules, $x = 2X_1/\sqrt{\pi}\, l_w$ (l_w: the mean free path of the gas molecules in the equilibrium state at rest with pressure p_w and temperature T_w; for a hard-sphere molecular gas, $l_w = mRT_w/\sqrt{2}\pi d_m^2 p_w$, where d_m is the diameter of a molecule), and $\hat{J}(\hat{\Phi}, \hat{\Phi})$ is the collision integral. The collision integral is expressed as

$$\hat{J}(\phi, \psi) = \frac{1}{2}\int (\phi'_*\psi' + \phi'\psi'_* - \phi_*\psi - \phi\psi_*)\hat{B}(|\,\boldsymbol{\alpha}\cdot(\,\boldsymbol{\zeta}_* - \,\boldsymbol{\zeta})|, |\,\boldsymbol{\zeta}_* - \,\boldsymbol{\zeta}|)\, d\Omega(\,\boldsymbol{\alpha})\, d\boldsymbol{\zeta}_*, \tag{2.2}$$

with

$$\phi = \phi(\boldsymbol{\zeta}), \quad \phi_* = \phi(\boldsymbol{\zeta}_*), \quad \phi' = \phi(\boldsymbol{\zeta}'), \quad \phi'_* = \phi(\boldsymbol{\zeta}'_*), \quad \text{etc.},$$
$$\boldsymbol{\zeta}' = \boldsymbol{\zeta} + \boldsymbol{\alpha}[\,\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})], \quad \boldsymbol{\zeta}'_* = \boldsymbol{\zeta}_* - \boldsymbol{\alpha}[\,\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})],$$
$$d\boldsymbol{\zeta}_* = d\zeta_{*1}\, d\zeta_{*2}\, d\zeta_{*3},$$

where $\hat{B}$ is a nonnegative function determined by the type of the intermolecular potential, $\boldsymbol{\alpha}$ is a unit vector, $d\Omega(\boldsymbol{\alpha})$ is the solid angle element in the direction of $\boldsymbol{\alpha}$, $\boldsymbol{\zeta}_*$ is the variable of integration corresponding to $\boldsymbol{\zeta}$, and the integration is carried out over the whole space of $\boldsymbol{\alpha}$ and that of $\boldsymbol{\zeta}_*$. For a hard-sphere molecular gas, the explicit form of the function $\hat{B}$ is given as

$$\hat{B} = |\boldsymbol{\alpha}\cdot(\boldsymbol{\zeta}_* - \boldsymbol{\zeta})|/4(2\pi)^{1/2}. \tag{2.3}$$

Further it should be noted that $\hat{B}$ for a gas with a general intermolecular potential depends on the parameter U_0/mRT_w, where U_0 is a characteristic magnitude of the intermolecular potential (Sone and Aoki, 1994).

The complete condensation condition on the condensed phase at $x = 0$ is given as

$$\hat{\Phi} = \pi^{-3/2}\exp(-\zeta_i^2), \quad (\zeta_1 > 0). \tag{2.4}$$

The condition at infinity, where a uniform flow is blowing, is given as

$$\hat{\Phi} = \hat{\Phi}_\infty = \frac{p_\infty/p_w}{\pi^{3/2}(T_\infty/T_w)^{5/2}} \exp[-(\zeta_i - \hat{u}_\infty \delta_{1i})^2/(T_\infty/T_w)]$$
$$= \frac{p_\infty/p_w}{\pi^{3/2}(T_\infty/T_w)^{5/2}} \exp[-(\zeta_i/(T_\infty/T_w)^{1/2} + \sqrt{5/6}M_\infty \delta_{1i})^2], \tag{2.5}$$

where $\hat{u}_\infty = v_\infty/(2RT_w)^{1/2}$ and $M_\infty = (3v_\infty^2/5RT_\infty)^{1/2}$ (the Mach number of the uniform flow).

The macroscopic variables of the gas, the density ρ or $\rho_w \hat{\omega}$ ($\rho_w = p_w/RT_w$), the flow velocity v_i or $(2RT_w)^{1/2}\hat{u}_i$ [$\hat{u}_i = (\hat{u}, 0, 0)$], the temperature T or $T_w \hat{\tau}$, the pressure p or $p_w \hat{p}$, etc., are expressed by the moments of $\hat{\Phi}$:

$$\hat{\omega} = \int \hat{\Phi}\, d\boldsymbol{\zeta}, \tag{2.6a}$$

$$\hat{\omega}\hat{u} = \int \zeta_1 \hat{\Phi}\, d\boldsymbol{\zeta}, \tag{2.6b}$$

$$\frac{3}{2}\hat{\omega}\hat{\tau} = \int (\zeta_j - \hat{u}\delta_{1j})^2 \hat{\Phi} d\boldsymbol{\zeta}, \tag{2.6c}$$

$$\hat{p} = \hat{\omega}\hat{\tau}, \tag{2.6d}$$

where the integration, and in what follows unless otherwise stated, is carried out over the whole space of $\boldsymbol{\zeta}$. The Mach number of the flow M [$= (v_i^2)^{1/2}(5RT/3)^{-1/2}$] is related to $\hat{u}$ by $M = \sqrt{6/5\hat{\tau}}|\hat{u}|$. For brevity in later description, we introduce the following notation:

$$\hat{\omega}_\infty = \rho_\infty/\rho_w, \quad \hat{u}_\infty = v_\infty/(2RT_w)^{1/2}, \quad \hat{p}_\infty = p_\infty/p_w, \quad \hat{\tau}_\infty = T_\infty/T_w. \tag{2.7}$$

For the Boltzmann-Krook-Welander model or BKW model (Bhatnagar *et al.*, 1954; Welander, 1954; Kogan, 1958), which will be used in the numerical computation of the present work, the collision integral $\hat{J}(\hat{\Phi}, \hat{\Phi})$ is expressed as

$$\hat{J}(\hat{\Phi}, \hat{\Phi}) = \hat{\omega}(\hat{\Phi}_e - \hat{\Phi}), \tag{2.8}$$

where $\hat{\Phi}_e$ is the local Maxwellian corresponding to $\hat{\Phi}$:

$$\hat{\Phi}_e = \frac{\hat{\omega}}{\pi^{3/2}\hat{\tau}^{3/2}} \exp\left(-\frac{(\zeta_i - \hat{u}\delta_{1i})^2}{\hat{\tau}}\right). \tag{2.9}$$

The boundary-value problem [(2.1), (2.4), and (2.5)] (or the half-space problem of condensation) is characterized by the four parameters Mach number M_∞, pressure ratio p_∞/p_w, temperature ratio T_∞/T_w, and relative size of the intermolecular potential U_0/mRT_w (by the first three parameters for a hard-sphere molecular gas and the BKW model). In the following, explicit analysis is carried out for a hard-sphere molecular gas or the BKW model. (The procedure of the analysis is applied for a gas with a general intermolecular potential.) Thus the Mach number M_∞, the pressure ratio p_∞/p_w, and the temperature ratio T_∞/T_w are the parameters of our concern.

The half-space problem does not necessarily have a solution for an arbitrary set of the three parameters. The range of the parameters that allows a solution has been studied numerically or approximately as mentioned in

Sec. 1. According to Sone *et al.* (1986), the solution exists in the following region (*Fig.* 1). For $M_\infty < 1$, the solution exists when the set of the three parameters lies on a surface in the $(M_\infty, p_\infty/p_w, T_\infty/T_w)$ space, or

$$p_\infty/p_w = F_s(M_\infty, T_\infty/T_w). \tag{2.10}$$

For $M_\infty > 1$, the solution exists in a three dimensional domain in the $(M_\infty, p_\infty/p_w, T_\infty/T_w)$ space or

$$p_\infty/p_w > F_b(M_\infty, T_\infty/T_w). \tag{2.11}$$

The boundary of the domain $p_\infty/p_w = F_b(M_\infty, T_\infty/T_w)$ is related to the surface $p_\infty/p_w = F_s(M_\infty, T_\infty/T_w)$. A point on the boundary surface $p_\infty/p_w = F_b(M_\infty, T_\infty/T_w)$ is related to some point on the surface (2.10) by the shock condition (or the Rankine-Hugoniot relation).

The solution of the half-space problem shows a striking feature across the sonic condition. The results, however, are numerical ones, and thus are of limited nature, although the results by Sone *et al.* (1986) and Aoki *et al.* (1990, 1991) are very detailed. In the present paper, we try to clarify the analytical structure of the solution in the transonic region. More precisely, let the curve $M_\infty = 1$, $p_\infty/p_w = F_s(1_-, T_\infty/T_w)$ in the $(M_\infty, p_\infty/p_w, T_\infty/T_w)$ space be called the B-curve; we investigate the solution when the upstream flow (the flow at infinity) is in a supersonic state in the neighbourhood of the B-curve, with a special interest in the

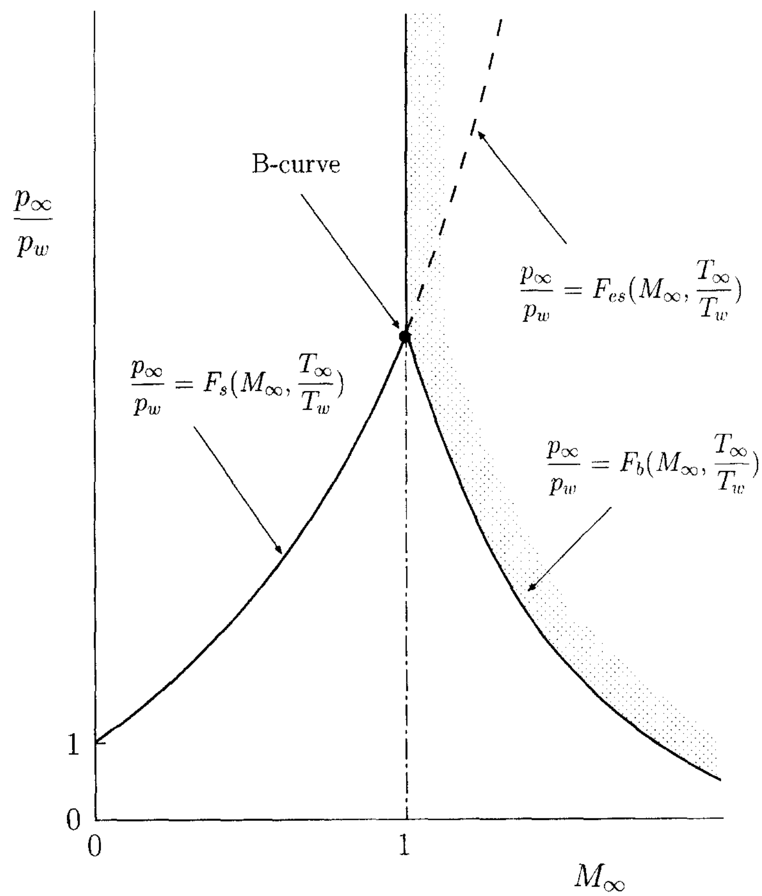

Fig. 1. – A schematic view of the section T_∞/T_w = const of the parameter space $(M_\infty, p_\infty/p_w, T_\infty/T_w)$ showing the range where a solution exists. A solution exists on $p_\infty/p_w = F_s(M_\infty, T_\infty/T_w)$ and $p_\infty/p_w > F_b(M_\infty, T_\infty/T_w)$. A supersonic Knudsen-layer-type solution (see Sec. 3.5) exists on $p_\infty/p_w = F_{es}(M_\infty, T_\infty/T_w)$.

abrupt change of the existence range of the solution across the sonic state. We also carry out detailed numerical computations in a region very close to the B-curve, which are not prepared in our previous work, and compare the analytical and numerical results.

3. Solution in a transonic range

3.1. Preliminaries

In the present paper, we mainly study the behaviour of the gas analytically when the upstream flow is on the supersonic side in the neighbourhood of the B-curve. The numerical confirmation that the boundary surface $p_\infty/p_w = F_b(M_\infty, T_\infty/T_w)$ is related to the surface $p_\infty/p_w = F_s(M_\infty, T_\infty/T_w)$ by the shock relation can also be well understood physically, and the behaviour of the gas near the surface may be considered to be roughly expressed by the combination of a shock wave and a subsonic solution (Sone *et al.*, 1986). In the neighbourhood of the B-curve, where $0 < M_\infty - 1 \ll 1$, the shock wave is weak and the quantities vary very slowly with distance (scaled with the mean free path). On the other hand, the subsonic solution varies appreciably over the distance of the mean free path and the variation is confined to a region of this distance. The solution of the half-space problem therefore has two distinct characters of the two solutions. Systematic analysis of the Boltzmann equation in this type of situation is introduced by Sone (1969, 1971) [see also Sone and Aoki (1994)]. We will make use of that method in the following analysis. First, a slowly varying solution perturbed slightly from a uniform state is discussed.

3.2. Slowly varying solution

Take a uniform equilibrium state with $\hat{u} = \hat{u}_U, \hat{\tau} = \hat{\tau}_U$, and $\hat{p} = \hat{p}_U$ (thus, $\hat{\omega} = \hat{\omega}_U = \hat{p}_U/\hat{\tau}_U$), where the nondimensional velocity distribution function $\hat{\Phi}_U$ is given as

$$\hat{\Phi}_U = \frac{\hat{p}_U}{\pi^{3/2}\hat{\tau}_U^{5/2}} \exp\left(-\frac{(\zeta_i - \hat{u}_U\delta_{1i})^2}{\hat{\tau}_U}\right). \tag{3.1}$$

In this subsection, we consider the case where the state of gas is slightly perturbed from the uniform state (3.1) and the perturbation is slowly varying with respect to the space variable x. By the "slowly varying", we mean that the velocity distribution function $\hat{\Phi}$ varies only slightly over the distance of $x = O(1)$, that is, $\partial\hat{\Phi}/\partial x = O(\varepsilon(\hat{\Phi} - \hat{\Phi}_U))$, where ε is the characteristic size of the perturbation from the uniform state. We will look for the solution of the Boltzmann equation (2.1) describing this situation. For the convenience of the following study, we introduce a shrunk coordinate X:

$$X = \varepsilon x. \tag{3.2}$$

Then the slowly varying condition is reduced to $\partial\hat{\Phi}/\partial X = O(\hat{\Phi} - \hat{\Phi}_U)$. With Eq. (3.2), the Boltzmann equation (2.1) is rewritten as follows:

$$\zeta_1 \frac{\partial\hat{\Phi}}{\partial X} = \frac{1}{\varepsilon}\hat{J}(\hat{\Phi}, \hat{\Phi}). \tag{3.3}$$

The slowly varying solution, indicated by the subscript S in the following, is to be obtained in the power series of ε:

$$\hat{\Phi}_S - \hat{\Phi}_U = \hat{\Phi}_{S1}\varepsilon + \hat{\Phi}_{S2}\varepsilon^2 + \cdots. \tag{3.4}$$

Correspondingly, the macroscopic variables $\hat{\omega}_S$, $\hat{u}_S$, $\hat{\tau}_S$, and $\hat{p}_S$ defined by Eqs. (2.6a)–(2.6d) with $\hat{\Phi} = \hat{\Phi}_S$ are also expanded as

(3.5) $$\hat{h}_S - \hat{h}_U = \hat{h}_{S1}\varepsilon + \hat{h}_{S2}\varepsilon^2 + \cdots,$$

where $\hat{h}_S$ represents any of the macroscopic variables $\hat{\omega}_S$, $\hat{u}_S$, $\hat{\tau}_S$, and $\hat{p}_S$; corresponding to these variables, $\hat{h}_U$ represents $\hat{\omega}_U$, $\hat{u}_U$, $\hat{\tau}_U$, and $\hat{p}_U$ respectively. By substitution of Eq. (3.4) into Eqs. (2.6a)–(2.6d), the component function $\hat{h}_{Sm}$ of $\hat{h}_S$ is expressed in terms of moments of $\hat{\Phi}_{Sn}$ $(n \leq m)$ [note the nonlinearity of Eqs. (2.6b)–(2.6d)]:

(3.6a) $$\hat{\omega}_{S1} = \int \hat{\Phi}_{S1} d\boldsymbol{\zeta},$$

(3.6b) $$\hat{\omega}_U \hat{u}_{S1} = \int (\zeta_1 - \hat{u}_U)\hat{\Phi}_{S1} d\boldsymbol{\zeta},$$

(3.6c) $$\frac{3}{2}\hat{\omega}_U \hat{\tau}_{S1} = \int [(\zeta_j - \hat{u}_U\delta_{1j})^2 - \frac{3}{2}\hat{\tau}_U]\hat{\Phi}_{S1} d\boldsymbol{\zeta},$$

(3.6d) $$\hat{p}_{S1} = \hat{\omega}_{S1}\hat{\tau}_U + \hat{\omega}_U\hat{\tau}_{S1},$$

(3.7a) $$\hat{\omega}_{S2} = \int \hat{\Phi}_{S2} d\boldsymbol{\zeta},$$

(3.7b) $$\hat{\omega}_U \hat{u}_{S2} = \int (\zeta_1 - \hat{u}_U)\hat{\Phi}_{S2} d\boldsymbol{\zeta} - \hat{\omega}_{S1}\hat{u}_{S1},$$

(3.7c) $$\frac{3}{2}\hat{\omega}_U \hat{\tau}_{S2} = \int [(\zeta_j - \hat{u}_U\delta_{1j})^2 - \frac{3}{2}\hat{\tau}_U]\hat{\Phi}_{S2} d\boldsymbol{\zeta} - \hat{\omega}_U\hat{u}_{S1}^2 - \frac{3}{2}\hat{\omega}_{S1}\hat{\tau}_{S1},$$

(3.7d) $$\hat{p}_{S2} = \hat{\omega}_{S2}\hat{\tau}_U + \hat{\omega}_U\hat{\tau}_{S2} + \hat{\omega}_{S1}\hat{\tau}_{S1},$$

....

Substituting Eq. (3.4) into Eq. (3.3), we obtain the following integral equations for the component functions $\hat{\Phi}_{Sm}$ $(m = 1, 2, \cdots)$ of the velocity distribution function $\hat{\Phi}_S$:

(3.8) $$\hat{J}(\hat{\Phi}_U, \hat{\Phi}_{S1}) = 0,$$

(3.9) $$2\hat{J}(\hat{\Phi}_U, \hat{\Phi}_{Sm}) = \zeta_1 \frac{\partial \hat{\Phi}_{Sm-1}}{\partial X} - \sum_{l=1}^{m-1} \hat{J}(\hat{\Phi}_{Sl}, \hat{\Phi}_{Sm-l}), \quad (m \geq 2).$$

The solution of the homogeneous linear integral equation (3.8) is given as (Grad, 1958; Cercignani, 1988)

(3.10) $$\hat{\Phi}_{S1} = \hat{\Phi}_U \left\{ \frac{\hat{\omega}_{S1}}{\hat{\omega}_U} + \frac{2\hat{u}_{S1}(\zeta_1 - \hat{u}_U)}{\hat{\tau}_U} + \frac{\hat{\tau}_{S1}}{\hat{\tau}_U}\left[\frac{(\zeta_j - \hat{u}_U\delta_{1j})^2}{\hat{\tau}_U} - \frac{3}{2} \right] \right\}.$$

Then, $\hat{\Phi}_U + \varepsilon\hat{\Phi}_{S1}$ is written as follows:

(3.11) $$\hat{\Phi}_U + \varepsilon\hat{\Phi}_{S1} = \frac{\hat{p}_U + \varepsilon\hat{p}_{S1}}{\pi^{3/2}(\hat{\tau}_U + \varepsilon\hat{\tau}_{S1})^{5/2}} \exp\left(-\frac{[\zeta_i - (\hat{u}_U + \varepsilon\hat{u}_{S1})\delta_{1i}]^2}{\hat{\tau}_U + \varepsilon\hat{\tau}_{S1}} \right) + O(\varepsilon^2).$$

The slowly varying solution of Eq. (2.1) is Maxwellian up to the order of ε. This property will be used in our later analysis.

Equation (3.9) is an inhomogeneous linear integral equation for $\hat{\Phi}_{Sm}$ $(m \geq 2)$. The homogeneous equation corresponding to Eq. (3.9), which is the same form as Eq. (3.8), has five nontrivial solutions $\hat{\Phi}_{U}$, $\hat{\Phi}_{U}\zeta_i$, and $\hat{\Phi}_{U}\zeta_j^2$. Thus, the solution of Eq. (3.9) is expressed in the form:

$$\hat{\Phi}_{Sm} = \hat{\Phi}_{U}(c_{m0} + c_{m1}\zeta_1 + c_{m4}\zeta_j^2) + \hat{\Psi}_{Sm}, \qquad (m \geq 2), \tag{3.12}$$

where the term proportional to ζ_2 or ζ_3 is omitted because of the symmetry of the problem, $\hat{\Psi}_{Sm}$ is the particular solution of Eq. (3.9) satisfying

$$\int (1, \zeta_i, \zeta_j^2)\hat{\Psi}_{Sm} d\boldsymbol{\zeta} = 0, \tag{3.13}$$

and c_{m0}, c_{m1}, and c_{m4} are undetermined functions of X and related to the macroscopic variables $\hat{\omega}_{Sm}$, $\hat{u}_{Sm}$, and $\hat{\tau}_{Sm}$ [these variables are determined by c_{n0}, c_{n1}, and c_{n4} $(n \leq m)$]. With these relations, the combination of c_{mi} $(m = 2)$ in the first term of Eq. (3.12) is expressed in the following form:

$$\begin{aligned} c_{20} + c_{21}\zeta_1 + c_{24}\zeta_j^2 = {} & \frac{\hat{\omega}_{S2}}{\hat{\omega}_{U}} + \frac{2(\zeta_1 - \hat{u}_{U})}{\hat{\tau}_{U}}\left[\hat{u}_{S2} + \frac{\hat{\omega}_{S1}\hat{u}_{S1}}{\hat{\omega}_{U}}\right] \\ & + \left[\frac{\hat{\tau}_{S2}}{\hat{\tau}_{U}} + \frac{2\hat{u}_{S1}^2}{3\hat{\tau}_{U}} + \frac{\hat{\omega}_{S1}\hat{\tau}_{S1}}{\hat{\omega}_{U}\hat{\tau}_{U}}\right]\left[\frac{(\zeta_j - \hat{u}_{U}\delta_{1j})^2}{\hat{\tau}_{U}} - \frac{3}{2}\right]. \end{aligned} \tag{3.14}$$

The inhomogeneous term (say, Ih_m) of Eq. (3.9) should satisfy the following solvability condition for Eq. (3.9) to have a solution, since the corresponding homogeneous equation has the nontrivial solutions mentioned above:

$$\int (1, \zeta_1, \zeta_j^2) Ih_m d\boldsymbol{\zeta} = \int (1, \zeta_1, \zeta_j^2)\zeta_1 \frac{\partial \hat{\Phi}_{Sm-1}}{\partial X} d\boldsymbol{\zeta} = 0, \quad (m \geq 2), \tag{3.15}$$

where the contribution of the collision integral vanishes.

From the solvability condition (3.15) with $m = 2$, we obtain the following homogeneous simultaneous linear equations for $d\hat{\omega}_{S1}/dX$, $d\hat{u}_{S1}/dX$, and $d\hat{\tau}_{S1}/dX$:

$$\begin{pmatrix} \hat{u}_{U} & \hat{\omega}_{U} & 0 \\ \hat{\tau}_{U}/2 & \hat{\omega}_{U}\hat{u}_{U} & \hat{\omega}_{U}/2 \\ 0 & 2\hat{\omega}_{U}\hat{u}_{U}^2 & 5\hat{\omega}_{U}\hat{u}_{U}/2 \end{pmatrix} \begin{pmatrix} d\hat{\omega}_{S1}/dX \\ d\hat{u}_{S1}/dX \\ d\hat{\tau}_{S1}/dX \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.16}$$

For the equations to have a nontrivial solution, the determinant composed of their coefficients should vanish, that is,

$$\hat{u}_{U} = 0, \quad \hat{u}_{U} = \pm(5\hat{\tau}_{U}/6)^{1/2}. \tag{3.17}$$

A slowly varying solution is possible only when the background uniform state is at rest or at a sonic state (Mach number = 1).

In the remaining part of this section, we study a slowly varying flow around a point on the B-curve (or one of the second case). [The first case is studied by Sone (1978).] Let the point concerned be denoted by the subscript B. Then,

$$\begin{aligned}
&\hat{u}_U = \hat{u}_B = -(5\hat{\tau}_B/6)^{1/2}, \ \hat{\tau}_U = \hat{\tau}_B,\\
&\hat{p}_U = \hat{p}_B = F_s(1_-, \hat{\tau}_B), \ \hat{\omega}_U = \hat{\omega}_B = \hat{p}_B/\hat{\tau}_B,\\
&\hat{\Phi}_U = \hat{\Phi}_B = \frac{\hat{p}_B}{\pi^{3/2}\hat{\tau}_B^{5/2}} \exp\left(-\frac{(\zeta_i - \hat{u}_B\delta_{1i})^2}{\hat{\tau}_B}\right).
\end{aligned} \tag{3.18}$$

The last relation on $\hat{p}_B$ is not used until the next section.

With this sonic condition as the background uniform state, we obtain the following relation from the system of equations (3.16):

$$\frac{d\hat{\omega}_{S1}}{dX} - \left(\frac{6}{5\hat{\tau}_B}\right)^{1/2} \hat{\omega}_B \frac{d\hat{u}_{S1}}{dX} = 0, \tag{3.19}$$

$$\frac{d\hat{\tau}_{S1}}{dX} - \left(\frac{8\hat{\tau}_B}{15}\right)^{1/2} \frac{d\hat{u}_{S1}}{dX} = 0. \tag{3.20}$$

These equations are independent of molecular models. When Eqs. (3.19) and (3.20) are satisfied, the particular solution $\hat{\Psi}_{S2}$ in Eq. (3.12) for a hard-sphere molecular gas is given as follows:

$$\begin{aligned}
\hat{\Psi}_{S2} = \hat{\Phi}_B \Bigg[&-\frac{\xi_1 A(\xi)}{\hat{\omega}_B\hat{\tau}_B} \frac{d\hat{\tau}_{S1}}{dX} - \frac{(\xi_1^2 - \xi^2/3)B(\xi)}{\hat{\omega}_B\hat{\tau}_B^{1/2}} \frac{d\hat{u}_{S1}}{dX}\\
&+2\left(\xi_1^2 - \frac{\xi^2}{3}\right)\frac{\hat{u}_{S1}^2}{\hat{\tau}_B} + 2\xi_1\left(\xi^2 - \frac{5}{2}\right)\frac{\hat{u}_{S1}\hat{\tau}_{S1}}{\hat{\tau}_B^{3/2}}\\
&+\left(\frac{1}{2}\xi^4 - \frac{5}{2}\xi^2 + \frac{15}{8}\right)\frac{\hat{\tau}_{S1}^2}{\hat{\tau}_B^2}\Bigg],
\end{aligned} \tag{3.21}$$

$$\xi_i = (\zeta_i - \hat{u}_B\delta_{1i})\hat{\tau}_B^{-1/2}, \qquad \xi = (\xi_j^2)^{1/2},$$

where $A(\xi)$ and $B(\xi)$ are the solutions of linear integral equations related to the collision integral. They have been studied by Pekeris and Alterman (1957) and Ohwada and Sone (1992). The functions $A(\xi)$ and $B(\xi)$ here are, respectively, the same functions as $A(\zeta)$ and $B(\zeta)$ in Ohwada and Sone (1992) and Sone *et al.* (1996), where their tables and figures are given; that is, they are obtained from $A(\zeta)$ and $B(\zeta)$ there simply by replacing ζ by ξ. [The function $\nu(\zeta)$ in Eqs. (A1)–(A5) on page 409 in Ohwada and Sone (1992) is a misprint for $2\sqrt{2}\nu(\zeta)$.] The solution $\hat{\Psi}_{S2}$ for the BKW equation is given simply by replacing $A(\xi)$ and $B(\xi)$ in Eq. (3.21) by $(\xi^2 - 5/2)\hat{\tau}_B^{1/2}$ and $2\hat{\tau}_B^{1/2}$ respectively.

From the solvability condition (3.15) with $m = 3$, we obtain the following linear differential equations for the $\hat{\omega}_{S2}$, $\hat{u}_{S2}$, and $\hat{\tau}_{S2}$:

$$\begin{pmatrix} \hat{u}_B & \hat{\omega}_B & 0 \\ \hat{\tau}_B/2 & \hat{\omega}_B\hat{u}_B & \hat{\omega}_B/2 \\ 0 & 2\hat{\omega}_B\hat{u}_B^2 & 5\hat{\omega}_B\hat{u}_B/2 \end{pmatrix} \begin{pmatrix} d\hat{\omega}_{S2}/dX \\ d\hat{u}_{S2}/dX \\ d\hat{\tau}_{S2}/dX \end{pmatrix} \tag{3.22}$$

$$= -\begin{pmatrix} \dfrac{d\hat{\omega}_{S1}\hat{u}_{S1}}{dX} \\ \dfrac{1}{2}\dfrac{d\hat{\omega}_{S1}\hat{\tau}_{S1}}{dX} + (\hat{\omega}_B\,\hat{u}_{S1} + \hat{\omega}_{S1}\hat{u}_B)\dfrac{d\hat{u}_{S1}}{dX} - \dfrac{2}{3}\gamma_1\hat{\tau}_B^{1/2}\dfrac{d^2\hat{u}_{S1}}{dX^2} \\ \hat{\omega}_B\hat{u}_B\dfrac{d\hat{u}_{S1}^2}{dX} - \dfrac{5}{4}\gamma_2\hat{\tau}_B^{1/2}\dfrac{d^2\hat{\tau}_{S1}}{dX^2} - \dfrac{4}{3}\gamma_1\hat{u}_B\hat{\tau}_B^{1/2}\dfrac{d^2\hat{u}_{S1}}{dX^2} \end{pmatrix},$$

where the inhomogeneous terms, which depend on molecular models, are for a hard-sphere molecular gas, and γ_1 and γ_2 are constants ($\gamma_1 = 1.270042427$ and $\gamma_2 = 1.922284066$) defined by the sixth order moments of B and A respectively (Ohwada and Sone, 1992; Sone *et al.*, 1996). They are, respectively, related to the viscosity and thermal conductivity of the gas. The homogeneous part of Eq. (3.22) is of the same form as Eq. (3.16), which is a homogeneous system with nontrivial solutions. Then the inhomogeneous terms of Eq. (3.22) should satisfy the solvability condition:

(3.23) $$\frac{d\hat{u}_{S1}^2}{dX} + \left(\frac{5}{6\hat{\tau}_B}\right)^{1/2}\hat{\tau}_{S1}\frac{d\hat{u}_{S1}}{dX} - \frac{1}{3}(2\gamma_1+\gamma_2)\hat{\omega}_B^{-1}\hat{\tau}_B^{1/2}\frac{d^2\hat{u}_{S1}}{dX^2} = 0.$$

The equations (3.19), (3.20), and (3.23) form the system of equations that determines $\hat{\omega}_{S1}$, $\hat{u}_{S1}$, and $\hat{\tau}_{S1}$. Incidentally, for the BKW equation only the third term of Eq. (3.23) is subject to modification, that is, $\gamma_1 = \gamma_2 = 1$, and $\hat{\tau}_B^{1/2}$ there should be replaced by $\hat{\tau}_B$.

We summarize the slowly varying solutions that approach a uniform state at positive infinity ($X = +\infty$) without their derivation, since Eqs. (3.19), (3.20), and (3.23) are easy to handle. The solutions consist of two families of solutions, supersonic at positive infinity ($M_\infty \geq 1$), in each of which the solutions are obtained by translation of the space coordinate X from a seed solution. This freedom plays an essential role in the abrupt change of the existence range of solution of the half-space problem. The seed solutions are given in the following form, where we do not discriminate quantities of difference of $O(\varepsilon^2)$ [*e.g.*, $\hat{\Phi}_B + \varepsilon\hat{\Phi}_{S1}$ and $\hat{p}_B + \varepsilon\hat{p}_{S1}$ are identified, respectively, with $\hat{\Phi}_S$ and $\hat{p}_S$], since only the quantities up to the order of ε (or the leading term of a nonuniform state) are of interest in the following analysis. The distribution function $\hat{\Phi}_S$ is a Maxwellian [see Eq. (3.11)]:

(3.24) $$\hat{\Phi}_S = \frac{\hat{p}_S}{\pi^{3/2}\hat{\tau}_S^{5/2}}\exp\left(-\frac{(\zeta_i - \hat{u}_S\delta_{1i})^2}{\hat{\tau}_S}\right).$$

Its parametric macroscopic variables $\hat{u}_S$, $\hat{\tau}_S$, and $\hat{p}_S$ and the local Mach number M_S $(= \sqrt{6/5}\,|\hat{u}_S|/\hat{\tau}_S^{1/2})$ are expressed with the aid of a common profile function $S(X)$ in the form:

(3.25a) $$\hat{u}_S = -\left(\frac{5\hat{\tau}_\infty}{6}\right)^{1/2}\left[M_\infty - \frac{3}{2}(M_\infty - 1)S(X)\right],$$

(3.25b) $$\hat{\tau}_S = \hat{\tau}_\infty[1 + (M_\infty - 1)S(X)],$$

(3.25c) $$\hat{p}_S = \hat{p}_\infty\left[1 + \frac{5}{2}(M_\infty - 1)S(X)\right],$$

(3.25d) $$M_S = M_\infty - 2(M_\infty - 1)S(X),$$

(3.25e) $$X = |M_\infty - 1|\,x,$$

where the profile function $S(X)$ is given by either of the following two functions:

(3.26a) $$S(X) = \left\{1 + \exp\left(\frac{\sqrt{30}\,\hat{\omega}_\infty}{2\gamma_1+\gamma_2}X\right)\right\}^{-1},\ (-\infty < X < \infty),$$

$$S(X) = -\left\{\exp\left(\frac{\sqrt{30}\,\hat{\omega}_\infty}{2\gamma_1+\gamma_2}X\right) - 1\right\}^{-1}, \quad (0 < X < \infty). \tag{3.26b}$$

The functions (3.26a) and (3.26b) are shown in Figure 2. The definite value of ε is related to the possible choice of $S(X)$; for example, if $S(X/2)/2$ is taken as new $S(X)$, then ε is doubled. Here, ε is $|M_\infty - 1|$.

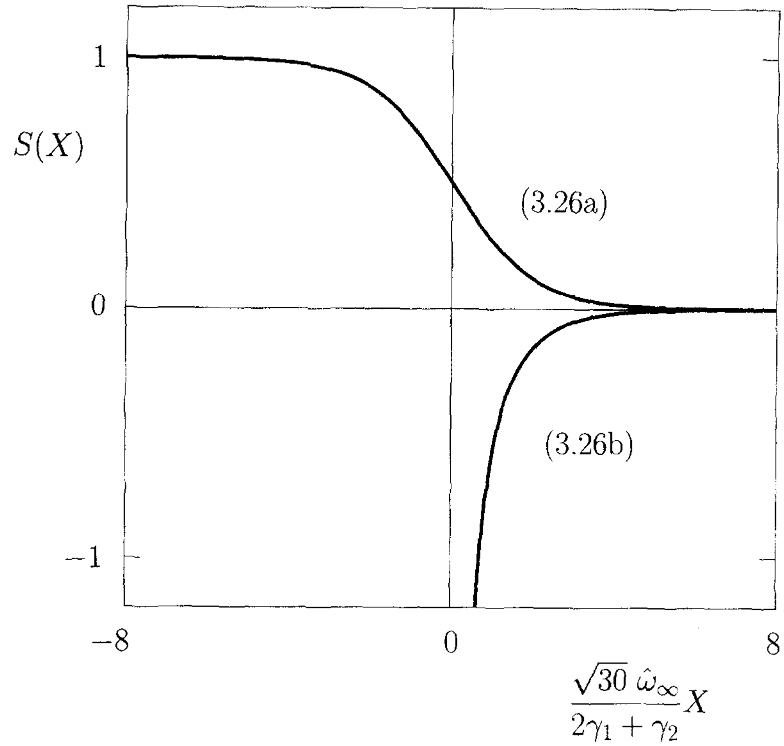

Fig. 2. – The functions $S(X)$ defined by Eqs. (3.26a) and (3.26b).
For the BKW equation, the abscissa should be replaced by $\sqrt{10/3}\,\hat{\omega}_\infty \hat{\tau}_\infty^{-1/2} X$.

At $M_\infty = 1$, only uniform flow ($\hat{u}_S < 0$) is possible. Equation (3.26a) corresponds to a weak shock wave ($\hat{u}_S < 0$), where M_S lies in the range $2 - M_\infty \leq M_S \leq M_\infty$. A weak shock wave as a solution of the Boltzmann equation is studied mathematically or analytically by Grad (1969), Cercignani (1970), Miyake (1980), Caflisch and Nicolaenko (1982), Ohwada (1993), etc. Equation (3.26b), limited to $X > 0$, corresponds to an accelerating flow ($\hat{u}_S < 0$) from a uniform supersonic flow. If the condition at positive infinity is discarded, a solution that approaches a uniform state at negative infinity is obtained. This is a subsonic accelerating flow approaching a uniform state at negative infinity and expressed by Eqs. (3.25a)–(3.25e), and (3.26b) with restriction for $M_\infty < 1$ and the replacement X by $-X$, where the subscript ∞ is taken to denote the value at negative infinity. The other solution, derived from $\hat{u}_{S1}$ that is defined in any finite interval and increases from $-\infty$ to $+\infty$, is an accelerating transonic flow valid only in a part of the finite range, the limitation of which is the requirement that the perturbation should be small. The slowly varying solution around the other sonic point $[\hat{u}_{L^-} = (5\hat{\tau}_{L^-}/6)^{1/2}]$ is obviously obtained by the reflection of the above four types of solution with respect to the space coordinate X. These additional solutions cannot of course be possible candidates as solutions to our half-space condensation problem. Incidentally, for the BKW equation we have only to modify $\hat{\omega}_\infty$, γ_1, and γ_2, respectively, by $\hat{\omega}_\infty \hat{\tau}_\infty^{-1/2}$, 1, and 1 in $S(X)$. It is noted that the variation of the slowly varying solution or the function S is characterized by the variable $(M_\infty - 1)X_1/l_\infty$, where l_∞ is the mean free path of the equilibrium state at rest with pressure p_∞ and temperature T_∞, in common for a hard-sphere molecular gas and the BKW model, since $l_w/l_\infty = \hat{\omega}_\infty$ (hard-sphere molecules) $= \hat{\omega}_\infty \hat{\tau}_\infty^{-1/2}$ (BKW).

The local Mach number M_S in these slowly varying solutions is a monotonic function of the position X, and the profile can be translated arbitrarily along X. Thus, the relation can be inverted (or X can be expressed by M_S), and any M_S between $2 - M_\infty$ and M_∞ for the shock wave and in $M_S \geq M_\infty$ for the supersonic accelerating flow can be made to correspond to the origin $X = 0$. From the relations (3.25b)–(3.25d) [or directly from Eqs. (3.19) and (3.20)], the pressure $\hat{p}_S$ and temperature $\hat{\tau}_S$ along the profile of any slowly varying flow are expressed in terms of the local Mach number M_S as follows:

$$\hat{p}_S = \hat{p}_\infty g_p(M_S, M_\infty), \quad \hat{\tau}_S = \hat{\tau}_\infty g_T(M_S, M_\infty), \tag{3.27}$$

where g_p and g_T are

$$g_p(M_S, M_\infty) = 1 + \frac{5}{4}(M_\infty - M_S), \quad g_T(M_S, M_\infty) = 1 + \frac{1}{2}(M_\infty - M_S). \tag{3.28}$$

These functions are independent of molecular models.

3.3. General description of solution of the half-space problem

Naturally the slowly varying solution does not satisfy the boundary condition (2.4) on the condensed phase unless $M_\infty = 0$, $T_\infty = T_w$, and $p_\infty = p_w$. We try to find the solution of the half-space problem by modifying the slowly varying function only in the neighbourhood of the condensed phase. That is, we put the solution in the form:

$$\hat{\Phi} = \hat{\Phi}_S + \hat{\Phi}_K, \tag{3.29}$$

where the correction term $\hat{\Phi}_K$ is assumed to vary appreciably over the distance of the order of the mean free path $[X_1 = O(l_w)$ or $x = O(1)]$ and to be appreciable only in the neighbourhood [or in $x = O(1)$] of the condensed phase, that is, $\partial\hat{\Phi}_K/\partial x = O(\hat{\Phi}_K)$, and $\hat{\Phi}_K$ vanishes very rapidly (or faster than any inverse power of x) as x tends to infinity (Knudsen-layer correction). We first rewrite Eq. (3.29) in the following form:

$$\hat{\Phi} = \hat{\Phi}_S - \hat{\Phi}_{e0} + \hat{\Phi}^*, \tag{3.30}$$

with

$$\hat{\Phi}_{e0} = \frac{(\hat{p}_S)_0}{\pi^{3/2}(\hat{\tau}_S{}^{5/2})_0} \exp\left(-\frac{[\zeta_i - (\hat{u}_S)_0\delta_{1i}]^2}{(\hat{\tau}_S)_0}\right), \tag{3.31}$$

$$\hat{\Phi}^* = \hat{\Phi}_{e0} + \hat{\Phi}_K, \tag{3.32}$$

where the quantities in the parentheses with subscript zero $(\ )_0$ are evaluated at $X = 0$, and $\hat{\Phi}^*$ is a rapidly varying function which deviates from $\hat{\Phi}_{e0}$ only in the neighbourhood of the condensed phase. It is noted that the Maxwellian $\hat{\Phi}_{e0}$ agrees with $(\hat{\Phi}_S)_0$ up to the order of ε. Substituting this form of $\hat{\Phi}$ into the Boltzmann equation (2.1), then we obtain the following equation:

$$\zeta_1 \frac{\partial\hat{\Phi}^*}{\partial x} = \hat{J}(\hat{\Phi}^*, \hat{\Phi}^*) + 2\hat{J}(\hat{\Phi}_S - \hat{\Phi}_{e0}, \hat{\Phi}_K). \tag{3.33}$$

In this derivation, we use the fact that $\hat{\Phi}_S$ satisfies Eq. (2.1) and that $\hat{J}(\hat{\Phi}_{e0}, \hat{\Phi}_{e0}) = 0$. In the second term on the right hand side of Eq. (3.33), a slowly varying function is multiplied by a rapidly decaying function, and therefore the former can be replaced by its series expansion:

$$\hat{\Phi}_S = \hat{\Phi}_B + \varepsilon(\hat{\Phi}_{S1})_0 + \varepsilon^2[(\hat{\Phi}_{S2})_0 + x(\partial\hat{\Phi}_{S1}/\partial X)_0] + \cdots. \tag{3.34}$$

From Eq. (3.34) and the fact that $\hat{\Phi}_S$ is Maxwellian up to the order of ε, the difference $\hat{\Phi}_S - \hat{\Phi}_{e0}$ is $O(\varepsilon^2)$. Thus we can estimate the second term on the right hand side of Eq. (3.33) as

$$\hat{J}(\hat{\Phi}_S - \hat{\Phi}_{e0}, \hat{\Phi}_K) = O(\varepsilon^2\hat{\Phi}_K). \tag{3.35}$$

It is confined to the region $[x = O(1)]$ where $\hat{\Phi}_K$ is appreciable, and is of the order of ε^2 there. Therefore, the rapidly varying function $\hat{\Phi}^*$, whose variation is confined to this region, is given within an error $O(\varepsilon^2)$ by the solution of the equation:

$$\zeta_1\frac{\partial\hat{\Phi}^*}{\partial x} = \hat{J}(\hat{\Phi}^*, \hat{\Phi}^*). \tag{3.36}$$

From Eqs. (2.4) and (3.30), at $x = 0$,

$$\hat{\Phi}^* = \pi^{-3/2}\exp(-\zeta_i^2) - [(\hat{\Phi}_S)_0 - \hat{\Phi}_{e0}], \quad (\zeta_1 > 0), \tag{3.37}$$

where $(\hat{\Phi}_S)_0 - \hat{\Phi}_{e0}$ is $O(\varepsilon^2)$. From Eq. (3.32) and the condition that $\hat{\Phi}_K \to 0$ rapidly as $x \to \infty$, the condition on $\hat{\Phi}^*$ at infinity is given by

$$\hat{\Phi}^* \to \frac{(\hat{p}_S)_0}{\pi^{3/2}(\hat{\tau}_S{}^{5/2})_0}\exp\left(-\frac{[\zeta_i - (\hat{u}_S)_0\delta_{1i}]^2}{(\hat{\tau}_S)_0}\right), \quad (\text{as } x \to \infty). \tag{3.38}$$

In the present study we are interested in the quantities up to the order of ε. Thus neglecting the differences of $O(\varepsilon^2)$, then we find that $\hat{\Phi}^*$ is determined by Eq. (3.36) with the boundary conditions (3.38) and (3.39) below:

$$\hat{\Phi}^* = \pi^{-3/2}\exp(-\zeta_i^2), \quad (\zeta_1 > 0;\ x = 0). \tag{3.39}$$

Hereafter, we use the same notation for the quantities with difference $O(\varepsilon^2)$, *e.g.*, $\hat{\Phi}_S = \hat{\Phi}_B + \varepsilon\hat{\Phi}_{S1}$, $(\hat{\Phi}_S)_0 = \hat{\Phi}_{e0}$, $\hat{u}_S = \hat{u}_B + \varepsilon\hat{u}_{S1}$, $\hat{p}_S = \hat{p}_B + \varepsilon\hat{p}_{S1}$, $\hat{\tau}_S = \hat{\tau}_B + \varepsilon\hat{\tau}_{S1}$. The error of the order of ε^2 is not only on $\hat{\Phi}^*$ but also on the relation among the parameters, *e.g.*, Eq. (2.10), that assures the existence of a solution. It is noted that $\hat{\Phi}^*$ at infinity in Eq. (3.38) is a Maxwellian. This condition on $\hat{\Phi}^*$ at infinity, which results from the fact that $\hat{\Phi}_S$ is Maxwellian (to the order of ε), is important in the following analysis.

The boundary-value problem [(3.36), (3.38), (3.39)] is apparently the same as the original half-space problem [(2.1), (2.4), (2.5)], with the replacement M_∞, p_∞/p_w, and T_∞/T_w by $(M_S)_0$, $(\hat{p}_S)_0$, and $(\hat{\tau}_S)_0$, and one may doubt the purpose of the preceding analysis. The points of difference are as follows. From the requirements on $\hat{\Phi}_K$, the new function $\hat{\Phi}^*$ approaches $(\hat{\Phi}_S)_0[= \hat{\Phi}_B + \varepsilon(\hat{\Phi}_{S1})_0]$ rapidly as $x \to \infty$, and its difference from $(\hat{\Phi}_S)_0$ is appreciable only in a region of $x = O(1)$ [not $X = O(1)$]. Instead of this restriction on $\hat{\Phi}^*$, the boundary condition (3.38) on $\hat{\Phi}^*$ at infinity has one free parameter, because, as shown in the previous subsection, $(M_S)_0$ in the slowly varying solution with a given set of M_∞, $\hat{p}_\infty$, and $\hat{\tau}_\infty$ can be chosen freely in the range $(M_S)_0 > 2 - M_\infty$. This freedom is related to the abrupt change of the existence range of a solution at the sonic point. For a given set of $M_\infty(> 1)$, $\hat{p}_\infty$, and $\hat{\tau}_\infty$, we can choose $(M_S)_0$ in such a way that the set of parameters $(M_S)_0$, $(\hat{p}_S)_0$, and $(\hat{\tau}_S)_0$ lies on the surface (2.10) [with the replacement

$M_\infty, p_\infty/p_w, T_\infty/T_w \to (M_S)_0, (\hat{p}_S)_0, (\hat{\tau}_S)_0]$. Admitting that a $\hat{\Phi}^*$-type subsonic solution exists on the surface, we have a supersonic solution in a domain of the parameter space $(M_\infty, p_\infty/p_w, T_\infty/T_w)$. This will be discussed in more detail with explicit examples in the following subsections.

3.4. Solution with a subsonic Knudsen layer

As explained in Sec. 2, the half-space problem for $M_\infty < 1$ has a solution only when the parameters M_∞, p_∞/p_w, and T_∞/T_w satisfy the condition $p_\infty/p_w = F_s(M_\infty, T_\infty/T_w)$. According to Sec. 3.2 , a slowly varying subsonic flow blowing against the condensed phase is impossible except in the case $M_\infty \ll 1$. Thus all the subsonic solutions for $|M_\infty - 1| \ll 1$ are of Knudsen-layer (or $\hat{\Phi}^*$) type. In this subsection we will discuss the solution of the half-space problem with subsonic $\hat{\Phi}^*$-type solution according to the general recipe described in the previous subsection.

First we supplement the data of the subsonic solution for $|M_\infty - 1| \ll 1$, obtained on the basis of the BKW equation. Figure 3 shows the surface $p_\infty/p_w = F_s(M_\infty, T_\infty/T_w)$ near the B-curve. The surface intersects the plane $M_\infty = 1$ at a finite angle. Figure 4 presents some examples of the profiles of macroscopic variables. The variables rapidly approach the values at infinity. In fact $\delta\bar{h}[= (h - h_\infty)/h_\infty]$, where h is p, T, v_1, or M and h_∞ is $p_\infty, T_\infty, v_\infty$, or M_∞ (don't confuse $\delta\bar{h}$ with $\hat{h}_S$ or $\hat{h}_L$ in Sec. 3.2), is less than 2.3×10^{-8} for $x \geq 2.5$ ($x \geq 6$ for the case f). Further, in order to examine the speed of decay of $\delta\bar{h}$ as $x \to \infty$, the function $\delta\bar{h}$ versus x and the derivative $d \ln |\ln |\delta\bar{h}(x)/\delta\bar{h}(0)||/d \ln x$ versus x for the case $M_\infty = 0.995$ and $T_\infty/T_w = 1$, for example, are shown for computations of different accuracies in Figure 5. The derivative becomes nearly uniform as x increases, but it changes sharply near the tail end. This sharp change, however, shifts to larger x and becomes moderate for more detailed computation. Thus, the sharp change is the false behaviour of

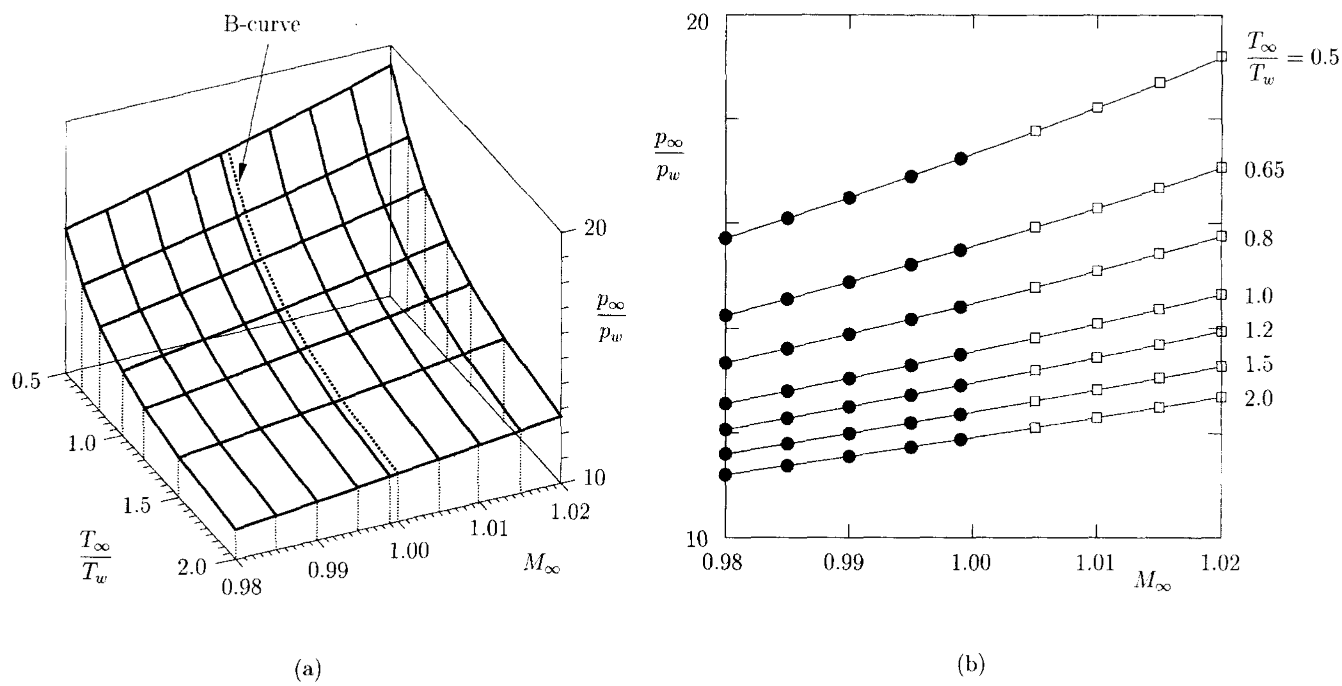

Fig. 3. – Numerical result for the two surfaces $p_\infty/p_w = F_s(M_\infty, T_\infty/T_w)$ [Eq. (3.40)] and $p_\infty/p_w = F_{ts}(M_\infty, T_\infty/T_w)$ [Eq. (3.46)] in the neighbourhood of the B-curve. (a) The two surfaces in the space $(M_\infty, p_\infty/p_w, T_\infty/T_w)$; (b) The sections $T_\infty/T_w = 0.5, 0.65, 0.8, 1, 1.2, 1.5$, and 2 of the two surfaces. In (b) the data (□) for F_{ts} lie on the extrapolated curve(——) of the data (•) of F_s. The two surfaces are joined smoothly at the sonic point (or on the B-curve).

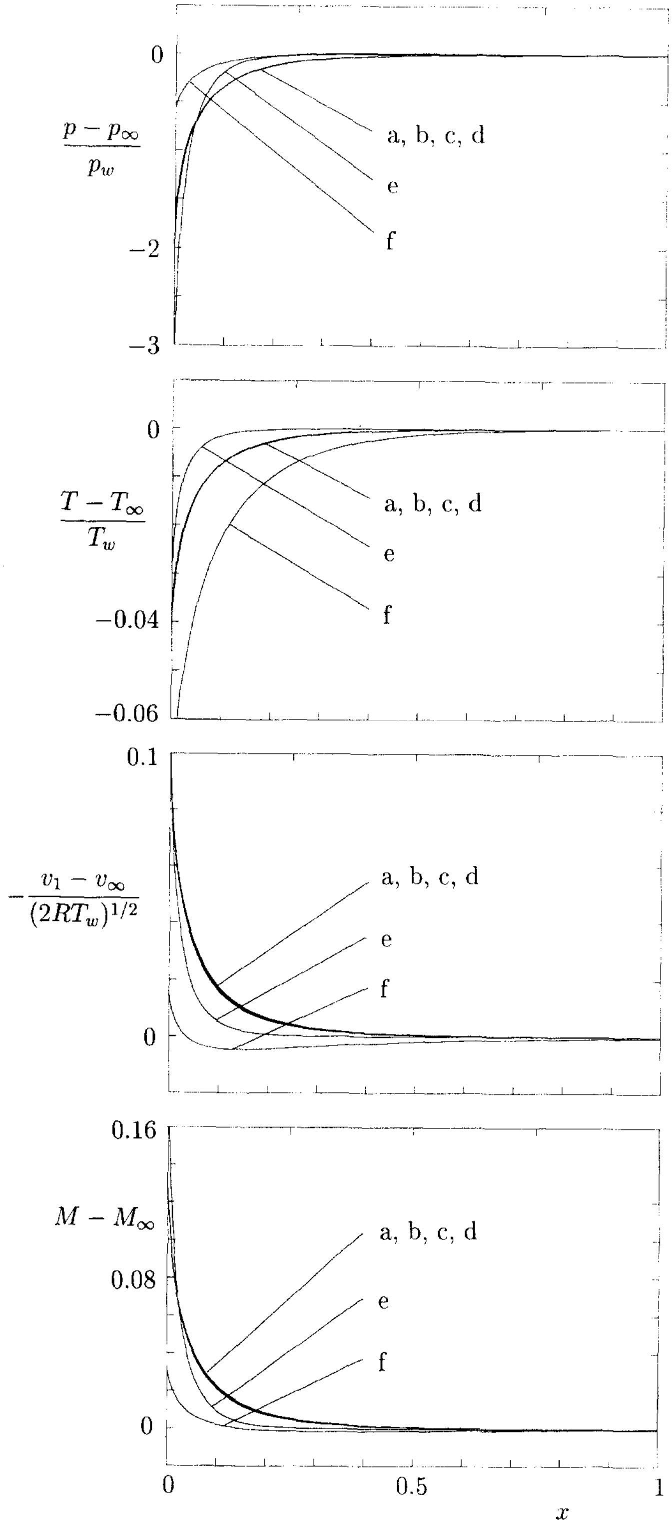

Fig. 4. – Numerical solutions of subsonic condensing flows ($M_\infty < 1$). All the solutions approach their uniform states rapidly as $x \to \infty$; that is, they are of Knudsen-layer type (or of $\hat{\Phi}^*$ type). The symbols a, b, c, d, e, and f correspond to the following sets of parameters (M_∞, p_∞/p_w, T_∞/T_w): a: (0.98, 12.563, 1); b: (0.99, 13.047, 1); c: (0.995, 13.298, 1); d: (0.999, 13.504, 1); e: (0.99, 16.484, 0.5); f: (0.99, 11.552, 2).

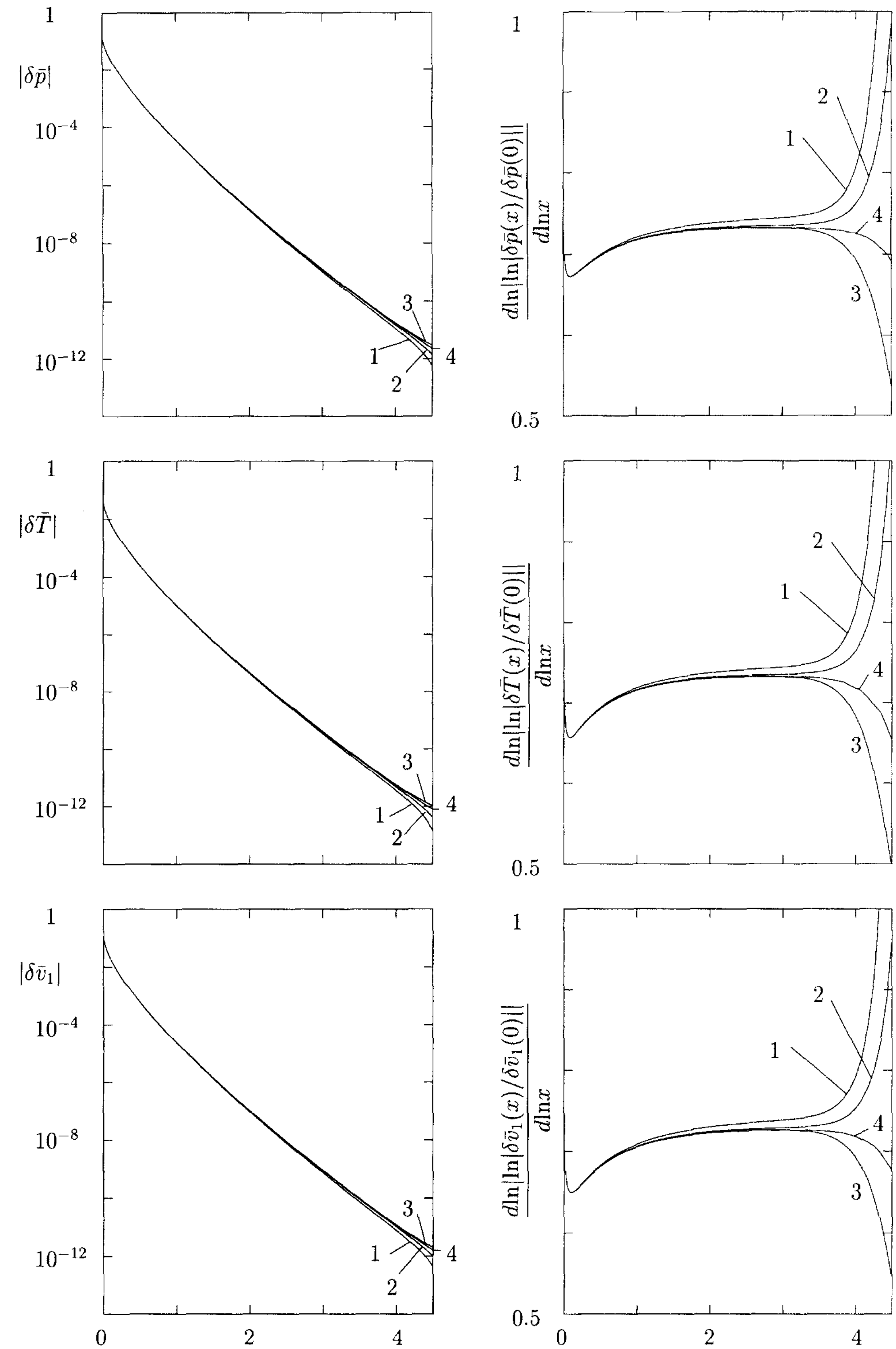

Fig. 5. – The function $\delta\bar{h}$ versus x and its derivative $d\ln|\ln|\delta\bar{h}(x)/\delta\bar{h}(0)||/d\ln x$ versus x for a subsonic Knudsen-layer-type solution for $M_\infty = 0.995$ and $T_\infty/T_w = 1$. The $\delta\bar{h} = (h - h_\infty)/h_\infty$, where h is p, T, or v_1 and h_∞ is p_∞, T_∞, or v_∞. (Do not confuse $\delta\bar{h}$ with $\hat{h}_S$ or $\hat{h}_{t'}$ in Sec. 3.2.) The numbers 1, 2, 3, and 4 indicate the computations with different accuracies; the lattice system is finer for larger number.

the derivative caused by the lack of accuracy of the computation for very small δh, and the derivative may be considered to be nearly uniform up to infinity and to approach a positive constant. Further, various tests for several Mach numbers close to unity show that the derivative is bounded below by a positive value with independent of M_∞. Then, in the neighbourhood of $M_\infty = 1$, δh decays faster than an exponential speed (or an exponential of some power of x) independent of M_∞. Therefore, the solution is surely of $\hat{\Phi}^*$ type. It is noted here that the variable x is related to X_1/l_∞, where l_∞ is the mean free path in the equilibrium state at rest with pressure p_∞ and temperature T_∞, by $X_1/l_\infty = \sqrt{\pi}(l_w/l_\infty)x/2$, *i.e.*, $X_1/l_\infty = \sqrt{\pi}(p_\infty/p_w)(T_w/T_\infty)x/2$ (hard-sphere molecules) $= \sqrt{\pi}(p_\infty/p_w)(T_w/T_\infty)^{3/2}x/2$ (BKW). In the present study we consider the case where the factors (p_∞/p_w) and (T_w/T_∞) are close to the values (p_B/p_w) and (T_w/T_B) of the point on the B-curve under consideration.

In the following analysis, on the basis of the numerical results, we assume that a subsonic solution, a solution of the half-space problem with $M_\infty < 1$, exists if and only if the parameters M_∞, p_∞/p_w, and T_∞/T_w satisfy the relation:

$$p_\infty/p_w = F_s(M_\infty, T_\infty/T_w). \tag{3.40}$$

(The mathematical proof is not given yet.) Further we assume that the surface intersects the plane $M_\infty = 1$ at a finite (or nonzero) angle and that there exists a solution of Knudsen-layer type on the intersection (or on the B-curve).

According to the discussion of the previous subsection, solution of the half-space problem for a given set of parameters M_∞ (> 1), p_∞/p_w, and T_∞/T_w is reduced to solution of the boundary-value problem for $\hat{\Phi}^*$ of Knudsen-layer type posed by Eqs. (3.36), (3.38), and (3.39). The set of parameters $(\hat{u}_S)_0$ [or $(M_S)_0$], $(\hat{p}_S)_0$, and $(\hat{\tau}_S)_0$, the values at the origin of the slowly varying solution, in the boundary condition (3.38) is a one-parameter set given by Eq. (3.27), *i.e.*,

$$(\hat{p}_S)_0 = \frac{p_\infty}{p_w} g_p((M_S)_0, M_\infty), \quad (\hat{\tau}_S)_0 = \frac{T_\infty}{T_w} g_T((M_S)_0, M_\infty), \tag{3.41}$$

where g_p and g_T are given by Eq. (3.28) and $(M_S)_0$ is taken as the free parameter. For the boundary-value problem to have a solution, this set of parameters should satisfy the condition (3.40), that is,

$$(\hat{p}_S)_0 = F_s((M_S)_0, (\hat{\tau}_S)_0), \quad [(M_S)_0 \leq 1]. \tag{3.42}$$

With the assumption (3.40), $(M_S)_0$ is limited to $(M_S)_0 \leq 1$. The slowly varying solution expressing a condensing flow ($\hat{u}_S < 0$) that can be subsonic is the weak shock wave, and its subsonic region ranges $2 - M_\infty \leq M_S \leq 1$. Thus, we can choose $(M_S)_0$ freely in the range $2 - M_\infty < (M_S)_0 \leq 1$, where the case $(M_S)_0 = 2 - M_\infty$ is excluded, since the varying part of the wave is then shifted to infinity. The $(\hat{p}_S)_0$ and $(\hat{\tau}_S)_0$ in Eq. (3.42) being given by Eq. (3.41) with $2 - M_\infty < M_S \leq 1$, the condition of existence of the solution of the half-space problem with subsonic Knudsen layer (or $\hat{\Phi}^*$) is given as follows:

$$\frac{p_\infty}{p_w} g_p(M_S, M_\infty) = F_s\left(M_S, \frac{T_\infty}{T_w} g_T(M_S, M_\infty)\right), \quad \text{with } 2 - M_\infty < M_S \leq 1, \tag{3.43}$$

where the parentheses with subscript 0, *i.e.*, $(\ )_0$, are omitted for simplicity. That is, a solution of the half-space problem with a subsonic Knudsen layer exists if and only if the set of the parameters M_∞, p_∞/p_w, and T_∞/T_w lies in the domain (3.43).

In the present work, where the case $0 < M_\infty - 1 \ll 1$ is considered, the formula (3.43) can be made more explicit. With the aid of Eq. (3.28), the existence range (3.43) is expressed as

(3.44)
$$\frac{p_\infty}{p_w} = F_s(1, \frac{T_\infty}{T_w}) - (M_\infty - 1)\Big(\frac{5}{2}F_s + \frac{\partial F_s}{\partial \eta_1} - \frac{T_\infty}{T_w}\frac{\partial F_s}{\partial \eta_2}\Big)_{(1, T_\infty/T_w)} + t\Big(\frac{5}{4}F_s + \frac{\partial F_s}{\partial \eta_1} - \frac{1}{2}\frac{T_\infty}{T_w}\frac{\partial F_s}{\partial \eta_2}\Big)_{(1, T_\infty/T_w)},$$
$$\text{for } 0 < t \leq M_\infty - 1, \quad (t = M_S + M_\infty - 2),$$

where η_1 and η_2 are, respectively, the first and second arguments of F_s, and the function and their derivatives are evaluated at $(\eta_1 = 1,\ \eta_2 = T_\infty/T_w)$. The formula (3.44) depends on molecular models only through F_s. For the BKW equation, all the six terms in the parentheses are positive, that is, the coefficients of $-(M_\infty - 1)$ and t are positive.

Let the set of the parameters M_∞ (> 1), p_∞/p_w, and T_∞/T_w be given. Let t (or M_S) computed by Eq. (3.44) be t_0 (or M_S^0). If it lies in $0 < t_0 \leq M_\infty - 1$ (or $2 - M_\infty < M_S^0 \leq 1$), there exists a solution of the half-space problem of condensation, which is expressed by a weak shock wave and a subsonic Knudsen layer. According to the argument in the previous subsection, the solution is constructed in the following way:

i) Compute $\hat{p}_S$ and $\hat{\tau}_S$ at $M_S = M_S^0$ by Eq. (3.41) with Eq. (3.28), and let them be $\hat{p}_S^0$ and $\hat{\tau}_S^0$. The set $(M_S^0,\ \hat{p}_S^0,\ \hat{\tau}_S^0)$ satisfies Eq. (3.42) by definition.

ii) Take the weak shock-wave solution, Eqs. (3.25a)–(3.25e) with Eq. (3.26a), satisfying $M_S = M_\infty$, $\hat{p}_S = p_\infty/p_w$, and $\hat{\tau}_S = T_\infty/T_w$ at $X = \infty$. Translate it in such a way that M_S takes the value M_S^0 at $X = 0$ [$S(X)$ in Eq. (3.26a) is replaced by $S(X + X_0)$]. Let it be $\hat{\Phi}_S^0$.

iii) Obtain the Knudsen-layer-type solution $\hat{\Phi}^*$ that approaches the Maxwellian distribution with M_S^0, $\hat{p}_S^0$, and $\hat{\tau}_S^0$ as $x \to \infty$. Its existence is assured, since the point $(M_S^0,\ \hat{p}_S^0,\ \hat{\tau}_S^0)$ lies on the surface $\hat{p}_S^0 = F_s(M_S^0, \hat{\tau}_S^0)$. Let it be $\hat{\Phi}_0^*$.

iv) The solution of the problem is given as

(3.45)
$$\hat{\Phi} = \hat{\Phi}_S^0 + [\hat{\Phi}_0^* - (\hat{\Phi}_S^0)_0],$$

where $(\hat{\Phi}_S^0)_0$ is $\hat{\Phi}_S^0$ at $X = 0$.

A supersonic solution, which exists in a domain in the parameter space $(M_\infty,\ p_\infty/p_w, T_\infty/T_w)$, is related to a subsonic Knudsen-layer-type solution on a surface in that space. Corresponding to this degeneration, a family of supersonic solutions can be generated from a subsonic solution. In fact, let the subsonic solution $\hat{\Phi}_0^*$ that takes $M = M^0$ (< 1), $\hat{\tau} = \hat{\tau}^0$, $\hat{p} = \hat{p}^0[= F_s(M^0, \hat{\tau}^0)]$ at $x = \infty$ be given. Take a family of weak shock-wave solutions $\hat{\Phi}_S^0$ that satisfy the conditions $M_S = M^0$ (< 1), $\hat{p}_S = \hat{p}^0$, $\hat{\tau}_S = \hat{\tau}^0$ at $X = 0$ and $M_S = M_\infty$ (> 1), $\hat{p}_S = \hat{p}^0/g_p(M^0, M_\infty)$, $\hat{\tau}_S = \hat{\tau}^0/g_T(M^0, M_\infty)$ at $X = \infty$, where M_∞ is the parameter in the range $M_\infty > 2 - M^0$. This family is obtained by translating the variable X of $S(X)$ in Eq. (3.26a). Then, from Eq. (3.45), a family of supersonic solutions that take $M = M_\infty$ (> 1), $\hat{p} = \hat{p}^0/g_p(M^0, M_\infty)$, $\hat{\tau} = \hat{\tau}^0/g_T(M^0, M_\infty)$ with the parameter M_∞ ranging $M_\infty > 2 - M^0$ at $X = \infty$ are generated from the single subsonic solution $\hat{\Phi}_0^*$. Figure 6 shows solutions generated from a subsonic Knudsen-layer-type solution for the BKW equation, where the new numerical solutions with the corresponding values of M_∞, p_∞/p_w, and T_∞/T_w at infinity are also shown for comparison. The analytical and numerical solutions agree very well.

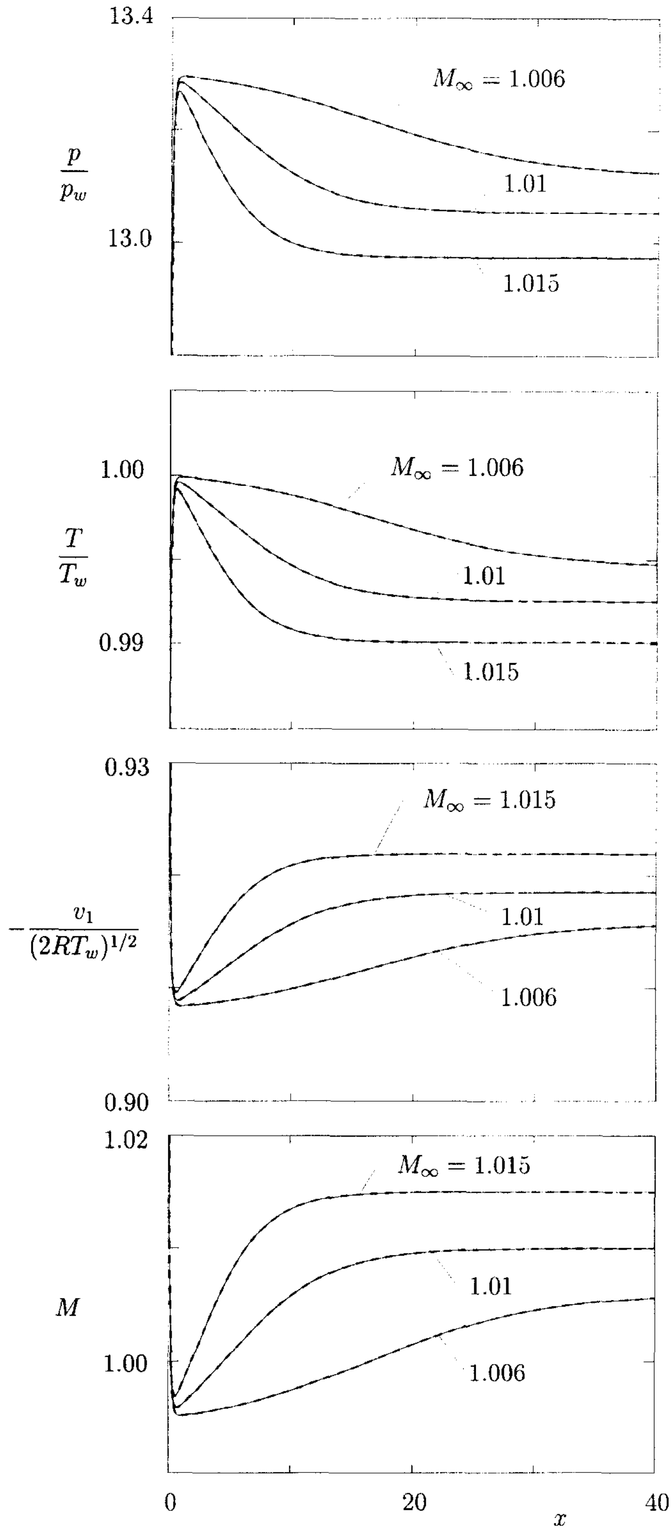

Fig. 6. – Various supersonic solutions generated from a subsonic Knudsen-layer-type solution $\hat{\Phi}^*$. In this example, $\hat{\Phi}^*$ that takes $M = 0.995$ and $\hat{\tau} = 1$ (thus $\hat{p} = 13.29786$) at $x = \infty$ are chosen. The supersonic solutions (——) are constructed by the recipe explained in the last paragraph in Sec. 3.4. The numerical solutions (-----) with the corresponding values of M_∞, p_∞/p_w, and T_∞/T_w at infinity are also shown for comparison.

3.5. SOLUTION WITH A SUPERSONIC KNUDSEN LAYER

In the previous subsection, we constructed a family of supersonic solutions from a subsonic Knudsen-layer-type solution with the aid of the slowly varying solution. The solutions, however, do not cover the whole neighbourhood of the B-curve ($M_\infty = 1$, $p_\infty = p_B$) where numerical solutions have been found. The slowly varying solution $\hat{\Phi}_S$ that was used in the analysis is a weak shock wave, and it is subsonic at $X = 0$ (in other words, the connection with the Knudsen layer was made only at a subsonic point on the weak shock wave). Thus we are naturally interested in the analytic structure of solutions in the remaining region in the neighbourhood of the B-curve and the possibility of construction of solutions with the aid of a supersonic part of the weak shock wave or a supersonic accelerating flow (the other slowly varying solution). In this subsection we will discuss these aspects.

Before starting the main discussion, we first review and supplement the numerical analysis of the problem for a better prospect of further study. In the paper by Sone *et al.* (1986), the steady solution of the half-space problem is studied as the limiting solution of a time-development problem, where the uniform state at infinity is taken as the initial state of the whole field. For a subsonic case, a compression wave (shock wave) propagates up to upstream infinity if $p_\infty/p_w < F_s$ and an expansion wave if $p_\infty/p_w > F_s$, and thus a steady state with a new uniform state at infinity is established. The new steady state lies on the surface $p_\infty/p_w = F_s(M_\infty, T_\infty/T_w)$. The feature of compression or expansion is inherited for a supersonic case, though the disturbance does not affect the state at infinity; that is, there is a compressed region in the front of the disturbance for p_∞/p_w smaller than some value, and not for larger p_∞/p_w. In the neighbourhood of the B-curve, the disturbance is extended far away from the condensed phase, but a solution of the Knudsen-layer type (or $\hat{\Phi}^*$ type) is expected to exist between the compression and expansion types of solution near a simple extrapolation of the $p_\infty/p_w = F_s$ surface. Then we can construct a family of supersonic solutions from this solution with the aid of the slowly varying solutions. Thus it is important here to confirm the existence of the supersonic solution of $\hat{\Phi}^*$ type. Here we give the numerical results for the BKW equation.

Figure 7 shows profiles of some macroscopic variables in the slightly supersonic region ($M_\infty = 1.01, T_\infty/T_w = 1$, and various p_∞/p_w). The solution is generally slowly varying, and the transition from compression to expansion type with increase of p_∞/p_w is clear. In between there seems to be a solution of $\hat{\Phi}^*$ type. To confirm the existence of this type of solution, we examine the variation of the variables with p_∞/p_w at various values of x (*Fig.* 8). For $x \geq 2.5$, the deviations from the values at infinity vanish at $p_\infty/p_w = 14.093889$ irrespective of x. More precisely, at $p_\infty/p_w = 14.093889$, the deviation of a variable from its value at infinity, *i.e.*, $\delta\bar{h} = (h - h_\infty)/h_\infty$, where h is p, T, v_1, or M and h_∞ is $p_\infty, T_\infty, v_\infty$, or M_∞, is less than 1.4×10^{-8} for $x \geq 2.5$. On the other hand, for p_∞/p_w =14.0 and 14.2, $\delta\bar{h}$ is, respectively, 8.3×10^{-4} and 8.7×10^{-4} at $x = 2.5$, 1.2×10^{-4} and 1.2×10^{-4} at $x = 10$, and 9.6×10^{-6} and 8.8×10^{-6} at $x = 20$. A small variation (about 1%) of p_∞/p_w gives a solution with a long tail. This distinct speed of variation for a special value of p_∞/p_w is clearly seen in the figure of $\delta\bar{h}$ versus $(M_\infty - 1)X_1/l_\infty$ with the ordinate in a logarithmic scale (*Fig.* 9), where $\delta\bar{h}$ decays with the same speed for all the other values of p_∞/p_w much slower than that for the special value. Thus the solution at $p_\infty/p_w = 14.093889$ may be considered of $\hat{\Phi}^*$ type. The above process of detecting a $\hat{\Phi}^*$-type solution is carried out for various M_∞ (close to unity) and T_∞/T_w. We thus find that the $\hat{\Phi}^*$-type solution exists on a surface $p_\infty/p_w = F_{es}(M_\infty, T_\infty/T_w)$, which is drawn in Figure 3. It is smoothly connected with $p_\infty/p_w = F_s(M_\infty, T_\infty/T_w)$. Several examples of the $\hat{\Phi}^*$-type supersonic solution thus obtained are given in Figure 10. The profiles are very close to those in Figure 4 for small difference of M_∞, which shows that the profiles vary smoothly across the B-curve and that the speed of approach to the uniform state is faster than some value independent of M_∞ in the neighbourhood of $M_\infty = 1$.

In order to examine the speed of approach to the uniform state of the $\hat{\Phi}^*$-type solution more closely, the function $\delta\bar{h}$ versus x and its slope $d\ln|\ln|\delta\bar{h}(x)/\delta\bar{h}(0)||/d\ln x$ versus x for the case $M_\infty = 1.01$ and

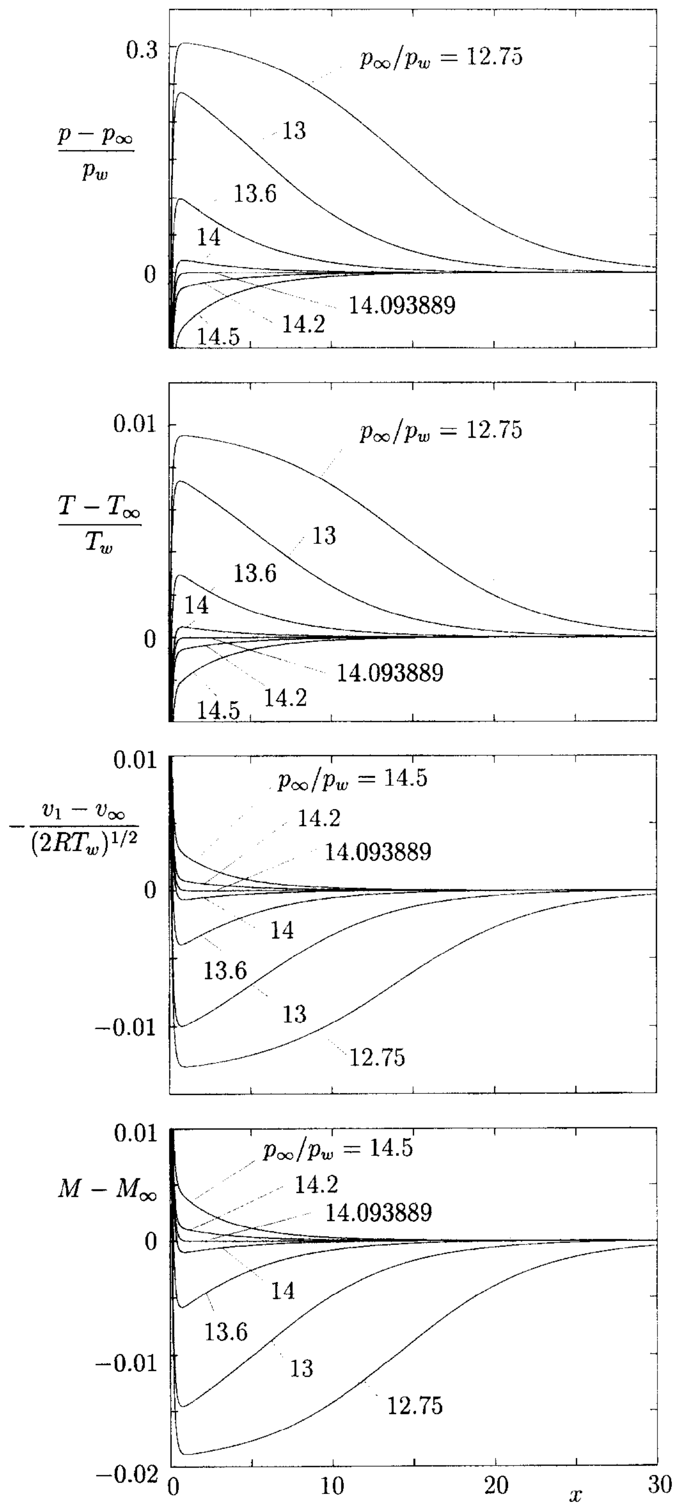

Fig. 7. – Numerical solutions of supersonic condensing flows ($M_\infty > 1$). Solutions for various p_∞/p_w at $M_\infty = 1.01$ and $T_\infty/T_w = 1$ are given. They have long tails (or slowly varying parts) except in the case $p_\infty/p_w = 14.093889$. There is a compression region in the front side for $p_\infty/p_w < 14.093889$.

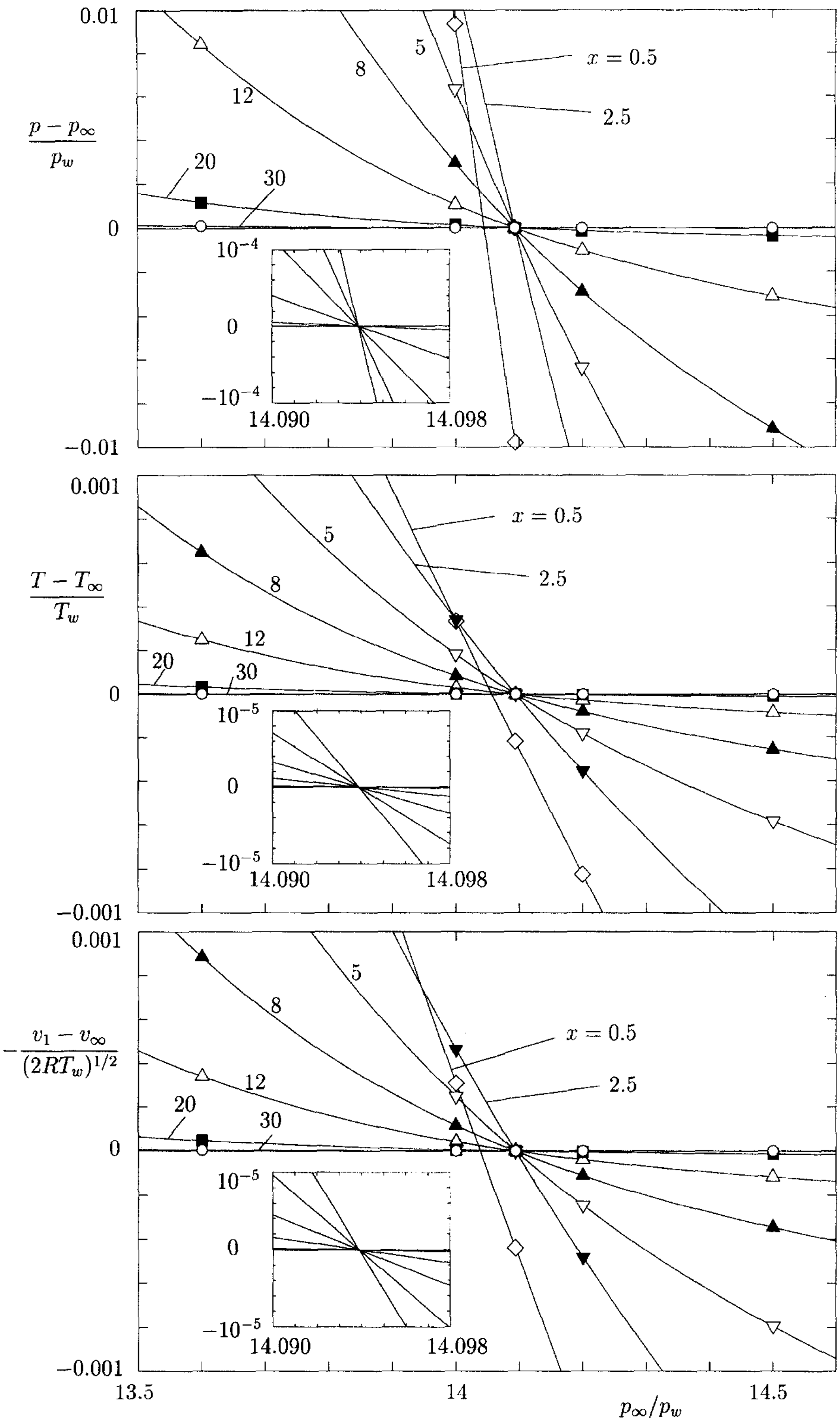

Fig. 8. – Numerical confirmation of existence of a supersonic Knudsen-layer-type solution. The variations of p , T, and v_1 with p_∞/p_w are shown for various point x ($M_\infty = 1.01$ and $T_\infty/T_w = 1$). For $x \geq 2.5$, the variables vanish at $p_\infty/p_w = 14.093889$. This means that the solution at $p_\infty/p_w = 14.093889$ is a solution of Knudsen-layer type.

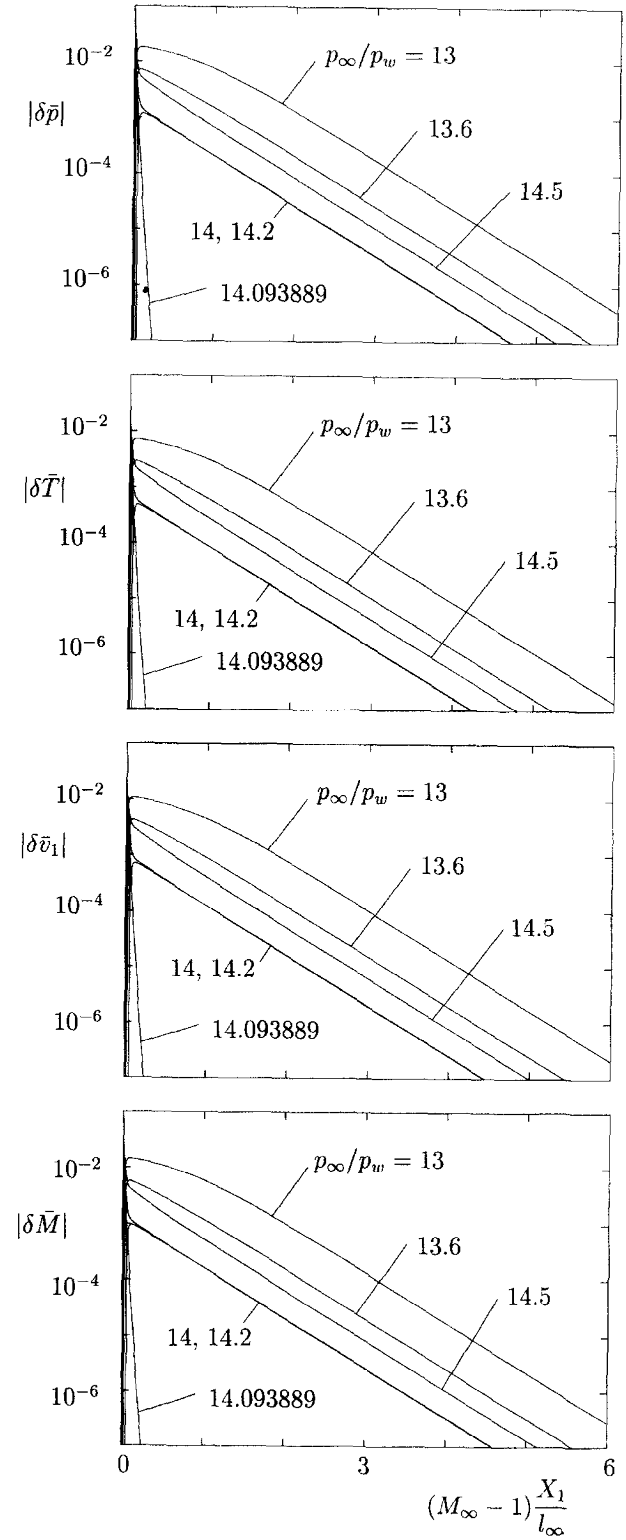

Fig. 9. – Comparison of the speed of variation of the supersonic solutions [$\delta\bar{h}$ versus $(M_\infty - 1)X_1/l_\infty$ with the ordinate in a logarithmic scale]. The variable $\delta\bar{h}$ decays exponentially with a common speed, in conformity with the behaviour of the slowly varying solution (3.25a)–(3.26b), except for $p_\infty/p_w = 14.093889$. At this p_∞/p_w, it decays with a distinctively much faster speed.

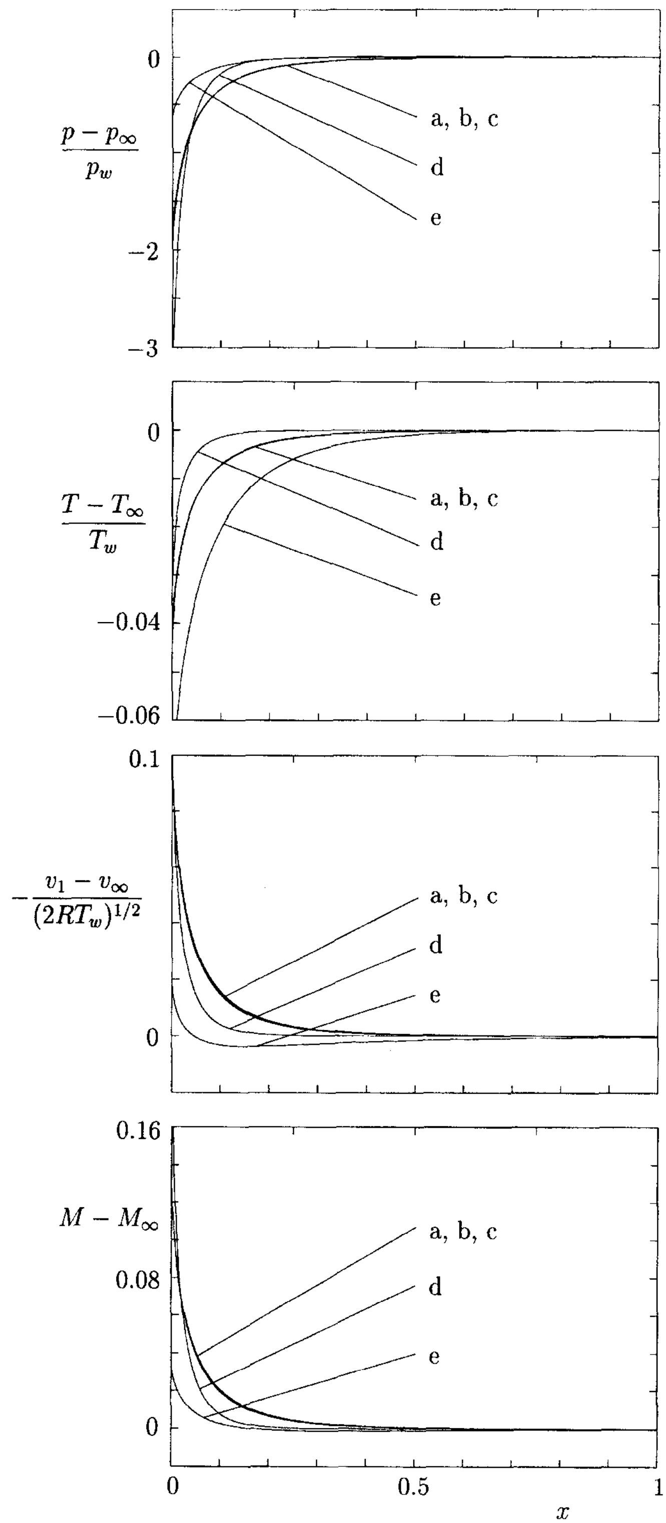

Fig. 10. – Numerical supersonic solutions of Knudsen-layer type on the surface $p_\infty/p_w = F_{cs}(M_\infty, T_\infty/T_w)$. The symbols a, b, c, d, and e correspond to the following sets of parameters (M_∞, p_∞/p_w, T_∞/T_w): a: (1.005, 13.8213, 1); b: (1.01, 14.0939, 1); c: (1.02, 14.6624, 1); d: (1.01, 18.2252, 0.5); e: (1.01, 12.3006, 2).

$T_\infty/T_w = 1$, as an example, are shown for computations of different accuracies in Figure 11. The slope becomes nearly uniform as x increases, but it decays sharply near the tail end. This decay region shifts to larger x, and the nearly uniform region extends farther for more detailed computation. Thus the sharp decay is the false behaviour of the slope caused by the lack of accuracy of the computation for very small $\delta\bar{h}$, and the derivative may be considered to be nearly uniform up to infinity and to approach a positive constant. Further, various tests for several Mach numbers close to unity show that the derivative is bounded below by a positive value independent of M_∞. Then, in the neighbourhood of M_∞, $\delta\bar{h}$ decays faster than with an exponential speed (or an exponential of some power of x) independent of M_∞. Therefore, the solution is surely of $\hat{\Phi}^*$ type. The sharp decay of the slope at the tail end is attributed to the error in determining the critical p_∞/p_w, owing to which a slight slowly varying part remains.

In the following analysis, on the basis of the above numerical investigation, we assume that a supersonic solution of Knudsen-layer type, a rapidly varying solution of the half-space problem with $M_\infty > 1$, exists if and only if the parameters M_∞, p_∞/p_w, and T_∞/T_w satisfy the relation:

$$p_\infty/p_w = F_{es}(M_\infty, T_\infty/T_w). \tag{3.46}$$

(The mathematical proof is not given yet.) Further the surface (3.46) is assumed to be smoothly joined with the surface (3.40) on the B-curve. Under that assumption we will construct the general supersonic solution.

There are two types of slowly varying solution that take supersonic speed at the origin ($X = 0$): one is the supersonic portion of the weak shock wave (or a decelerating flow) and the other is the accelerating flow discussed in Sec. 3.2. On both of these branches,

$$\hat{p}_S = \frac{p_\infty}{p_w} g_p(M_S, M_\infty), \quad \hat{\tau}_S = \frac{T_\infty}{T_w} g_T(M_S, M_\infty), \tag{3.47}$$

which is the same relation as in the subsonic case. The supersonic part of the weak shock wave corresponds to $1 < M_S \le M_\infty$, and the supersonic accelerating flow corresponds to $M_S > M_\infty$. (At $M_S = M_\infty$, the slowly varying flow degenerates into a uniform flow.) By the same argument in the previous subsection, the condition of existence of a solution of the half-space problem with a supersonic Knudsen layer is given as follows:

Corresponding to the weak shock wave,

$$\frac{p_\infty}{p_w} g_p(M_S, M_\infty) = F_{es}(M_S, \frac{T_\infty}{T_w} g_T(M_S, M_\infty)), \quad \text{for } 1 < M_S \le M_\infty, \tag{3.48}$$

and corresponding to the supersonic accelerating flow,

$$\frac{p_\infty}{p_w} g_p(M_S, M_\infty) = F_{es}(M_S, \frac{T_\infty}{T_w} g_T(M_S, M_\infty)), \quad \text{for } M_S > M_\infty. \tag{3.49}$$

The two relations (3.48) and (3.49) can be combined into one:

$$\frac{p_\infty}{p_w} g_p(M_S, M_\infty) = F_{es}(M_S, \frac{T_\infty}{T_w} g_T(M_S, M_\infty)), \quad \text{for } M_S > 1. \tag{3.50}$$

These formulae are simplified using the condition $0 < M_\infty - 1 \ll 1$ as follows:

$$\begin{aligned}\frac{p_\infty}{p_w} = F_s(1, \frac{T_\infty}{T_w}) &- (M_\infty - 1)\left(\frac{5}{2}F_s + \frac{\partial F_s}{\partial \eta_1} - \frac{T_\infty}{T_w}\frac{\partial F_s}{\partial \eta_2}\right)_{(1, T_\infty/T_w)} \\ &+ t\left(\frac{5}{4}F_s + \frac{\partial F_s}{\partial \eta_1} - \frac{1}{2}\frac{T_\infty}{T_w}\frac{\partial F_s}{\partial \eta_2}\right)_{(1, T_\infty/T_w)}, \\ &\text{for } t > M_\infty - 1, \ (t = M_S + M_\infty - 2),\end{aligned} \tag{3.51}$$

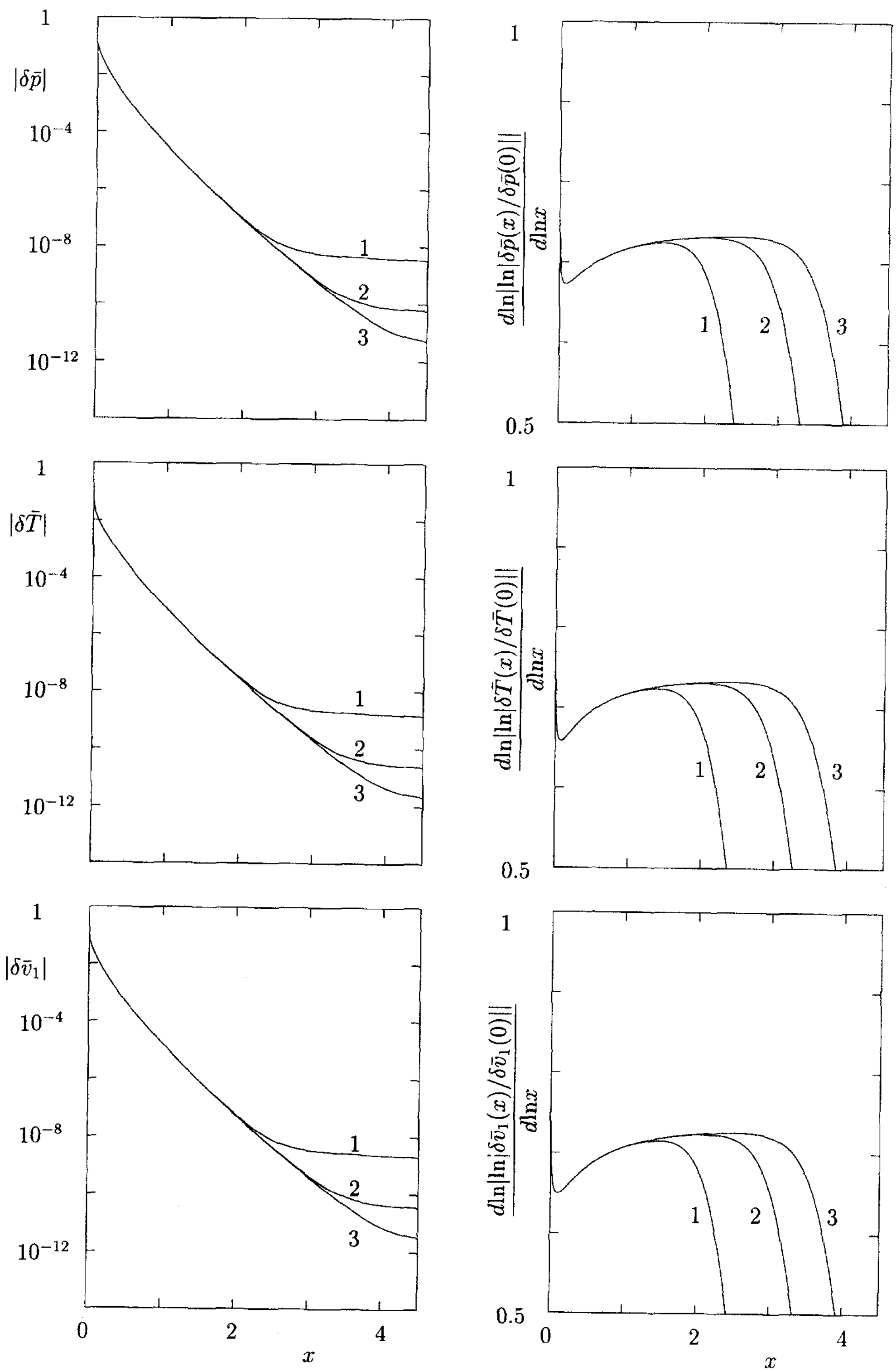

Fig. 11. – The function $\delta\bar{h}$ versus x and its slope $d\ln|\ln|\delta\bar{h}(x)/\delta\bar{h}(0)||/d\ln x$ versus x for a supersonic Knudsen-layer-type solution for $M_\infty = 1.01$ and $T_\infty/T_w = 1$. The $\delta\bar{h} = (h - h_\infty)/h_\infty$, where h is p, T, or v_1 and h_∞ is p_∞, T_∞, or v_∞. (Don't confuse $\delta\bar{h}$ with $\hat{h}_S$ or $\hat{h}_U$ in Sec. 3.2.) The numbers 1, 2, and 3 indicate the computations with different accuracies; the lattice system is finer for larger number.

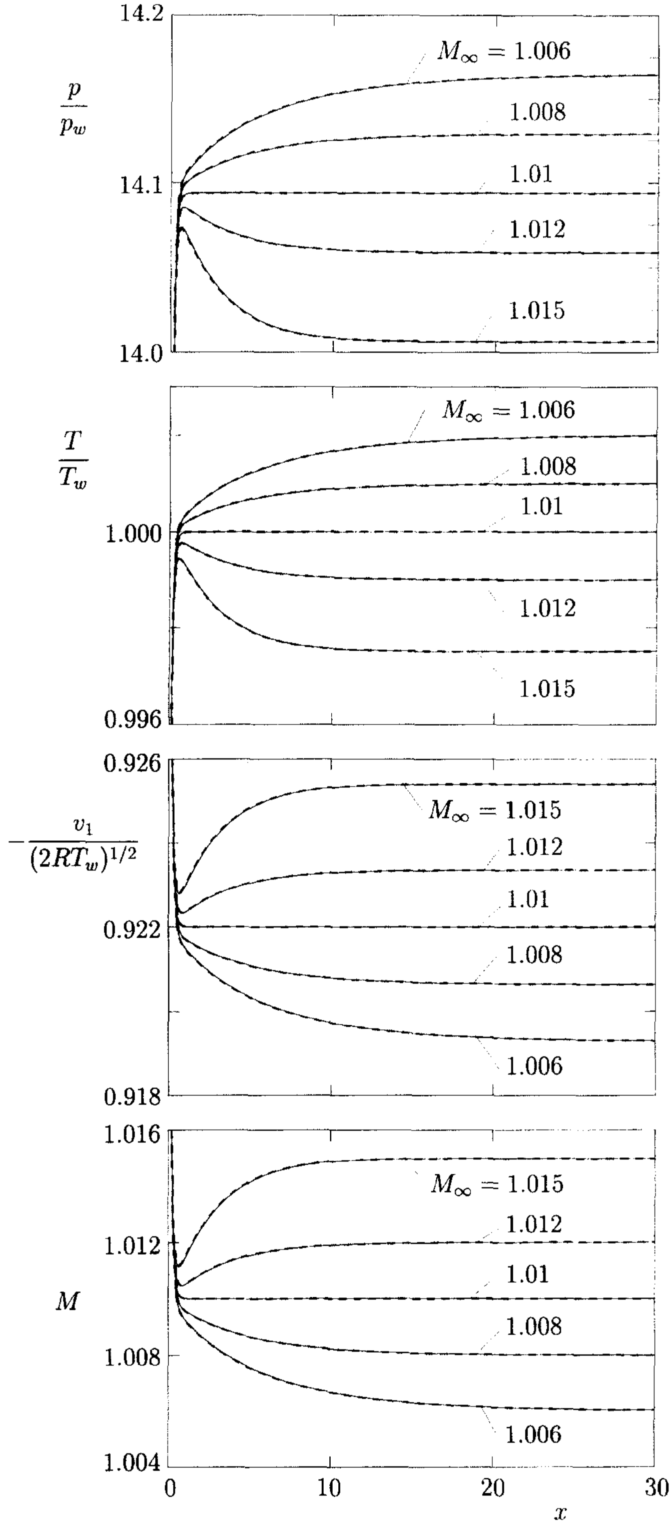

Fig. 12. – Various supersonic solutions generated from a supersonic Knudsen-layer-type solution $\hat{\Phi}^*$. In this example, $\hat{\Phi}^*$ that takes $M = 1.01$ and $\hat{\tau} = 1$ (thus $\hat{p} = 14.093889$) at $x = \infty$ are chosen. The supersonic solutions (——) are constructed by the recipe explained in the last paragraph in Sec. 3.5. The numerical solutions (-----) with the corresponding values of M_∞, p_∞/p_w, and T_∞/T_w at infinity are also shown for comparison. For the present parameters, the slowly varying part of the solutions is an accelerating flow if $M_\infty < 1.01$ and a decelerating flow (a part of a weak shock wave) if $M_\infty > 1.01$.

where the assumption that the two surfaces $p_\infty/p_w = F_s(M_\infty, T_\infty/T_w)$ and $p_\infty/p_w = F_{es}(M_\infty, T_\infty/T_w)$ are smoothly joined at $M_\infty = 1$ (or on the B-curve) is used.

Let the set of the parameters M_∞ (> 1), p_∞/p_w, and T_∞/T_w be given. Let t (or M_S) computed by Eq. (3.51) be t_0 (or M_S^0). If it lies in $t_0 > M_\infty - 1$ (or $M_S^0 > 1$), there exists a solution of the half-space problem of condensation, which is expressed by a weak shock wave or a supersonic accelerating flow and a supersonic Knudsen layer. The solution is constructed in the following way:

i) Compute $\hat{p}_S$ and $\hat{\tau}_S$ at $M_S = M_S^0$ by Eq. (3.41) with Eq. (3.28), and let them be $\hat{p}_S^0$ and $\hat{\tau}_S^0$. The set (M_S^0, $\hat{p}_S^0$, $\hat{\tau}_S^0$) satisfies $\hat{p}_S^0 = F_{es}(M_S^0, \hat{\tau}_S^0)$, corresponding to Eq. (3.46), by definition.

ii) Take a weak shock-wave solution if $M_\infty - 1 < t_0 \leq 2(M_\infty - 1)$, or a supersonic accelerating flow if $t_0 > 2(M_\infty - 1)$, satisfying $M_S = M_\infty$, $\hat{p}_S = p_\infty/p_w$, and $\hat{\tau}_S = T_\infty/T_w$ at $X = \infty$. Translate it in such a way that M_S takes the value M_S^0 at $X = 0$ [$S(X)$ in Eqs. (3.26a) and (3.26b) is replaced by $S(X+X_0)$]. Let it be $\hat{\Phi}_S^0$.

iii) Obtain the Knudsen-layer-type solution $\hat{\Phi}^*$ that approaches the Maxwellian distribution with M_S^0, $\hat{p}_S^0$, and $\hat{\tau}_S^0$ as $x \to \infty$. Its existence is assured, since the point (M_S^0, $\hat{p}_S^0$, $\hat{\tau}_S^0$) lies on the surface $\hat{p}_S^0 = F_{es}(M_S^0, \hat{\tau}_S^0)$. Let it be $\hat{\Phi}_0^*$.

iv) The solution of the problem is given as

$$\hat{\Phi} = \hat{\Phi}_S^0 + [\hat{\Phi}_0^* - (\hat{\Phi}_S^0)_0], \tag{3.52}$$

where $(\hat{\Phi}_S^0)_0$ is $\hat{\Phi}_S^0$ at $X = 0$. When $t_0 = 2(M_\infty - 1)$, the shock wave degenerates into a uniform flow and the solution is expressed only by $\hat{\Phi}_0^*$.

It should be noted that the slowly varying solution obtained in Sec. 3.2 violates the assumption of "slowly varying" when $M_\infty - 1 \ll (p_\infty - p_B)/p_B$ with $p_B/p_w = F_s(1, T_\infty/T_w)$. In this case, $t_0 \gg M_\infty - 1$ (or $M_S^0 - 1 \gg M_\infty - 1$). In view of Eq. (3.25d), $(M_\infty - 1)S(X + X_0) = O(M_S^0 - 1)$ or $S(X + X_0) = O[(M_S^0 - 1)/(M_\infty - 1)]$, which is very large. Thus, the real origin corresponds to a point very close to the origin of $S(X)$ in Eq. (3.26b), where $S(X)$ and its derivative are infinite. The present analysis is invalid in the thin wedge region $M_\infty - 1 \ll (p_\infty - p_B)/p_B$.

A supersonic solution, which exists in a domain in the parameter space (M_∞, p_∞/p_w, T_∞/T_w), is related to a supersonic solution of Knudsen-layer type on a surface in that space. Corresponding to this degeneration, a family of supersonic solutions that take $M = M_\infty$, $\hat{p} = \hat{p}^0/g_p(M^0, M_\infty)$, $\hat{\tau} = \hat{\tau}^0/g_T(M^0, M_\infty)$ at $X = \infty$ with the parameter M_∞ ranging $M_\infty > 1$ are generated from the supersonic Knudsen-layer-type solution $\hat{\Phi}_0^*$ that takes $M = M^0$ (> 1), $\hat{\tau} = \hat{\tau}^0$, $\hat{p} = \hat{p}^0$ [$= F_{es}(M^0, \hat{\tau}^0)$] at $x = \infty$. The slowly varying part of the solution is a supersonic part of a weak shock wave if $M_\infty > M^0$, and it is a supersonic accelerating flow if $M_\infty < M^0$ (the slowly varying part degenerates if $M_\infty = M^0$). Figure 12 is an example of various solutions generated from a supersonic Knudsen-layer-type solution for the BKW equation, where the new numerical solutions with the corresponding values of M_∞, p_∞/p_w , and T_∞/T_w at infinity are also shown for comparison. The analytical and numerical solutions agree very well.

3.6 Summary of the results

Combining the two results (3.44) and (3.51), one for the solution with a subsonic Knudsen layer and the other for the solution with a supersonic Knudsen layer, we find that the supersonic solution of the half-space problem of condensation exists if and only if the parameters M_∞, p_∞, and T_∞/T_w satisfy the relation:

$$\frac{p_\infty}{p_w} = F_s(1, \frac{T_\infty}{T_w}) - (M_\infty - 1)\left(\frac{5}{2}F_s + \frac{\partial F_s}{\partial \eta_1} - \frac{T_\infty}{T_w}\frac{\partial F_s}{\partial \eta_2}\right)_{(1, T_\infty/T_w)} \tag{3.53}$$

$$+\, t\left(\frac{5}{4}F_s + \frac{\partial F_s}{\partial \eta_1} - \frac{1}{2}\frac{T_\infty}{T_w}\frac{\partial F_s}{\partial \eta_2}\right)_{(1,\,T_\infty/T_w)}, \quad \text{for } t > 0.$$

In the formula there is no upper bound for t, but it should be noted that the present analysis is valid only when M_∞ and p_∞ are in the neighbourhood of the B-curve.

Let the set of the parameters $M_\infty\ (>1), T_\infty/T_w$, and p_∞/p_w be given. Depending on t, which is computed from Eq. (3.53), the solutions of the half-space problem of condensation are classified as follows:

i) There is no solution if $t \le 0$.

ii) The solution is expressed by a weak shock wave and a subsonic Knudsen layer if $0 < t < M_\infty - 1$.

iia) The solution is expressed by a weak shock wave and a sonic Knudsen layer if $t = M_\infty - 1$.

iii) The solution is expressed by a weak shock wave and a supersonic Knudsen layer if $M_\infty - 1 < t < 2(M_\infty - 1)$.

iiia) The solution is expressed only by a supersonic Knudsen layer if $t = 2(M_\infty - 1)$.

iv) The solution is expressed by an accelerating supersonic flow and a supersonic Knudsen layer if $t > 2(M_\infty - 1)$.

Thus we find the behaviour of the solution in the complete supersonic neighbourhood of the B-curve except in the thin wedge region $M_\infty - 1 \ll (p_\infty - p_B)/p_B$.

In the derivation of Eq. (3.53), it is assumed that two surfaces $p_\infty/p_w = F_s(M_\infty, T_\infty/T_w)$ and $p_\infty/p_w = F_{es}(M_\infty, T_\infty/T_w)$ are smoothly joined on the B-curve. If the connection is not smooth, there is a gap between or an overlapping of the two ranges (3.44) and (3.51), since $\partial F_s/\partial\eta_1$ and $\partial F_s/\partial\eta_2$ in the latter formula should be replaced by original $\partial F_{es}/\partial\eta_1$ and $\partial F_{es}/\partial\eta_2$, where the first and second arguments of F_{es} are also denoted, respectively, by η_1 and η_2. If the overlapping occurs, two solutions exist for a given set of parameters $(M_\infty,\ p_\infty/p_w, T_\infty/T_w)$ in the overlapping region.

4. Concluding remarks

The half-space problem of condensation, where a steady uniform flow is blowing against its plane condensed phase, is considered. The striking feature of the flow, first found numerically, that the range of the parameters M_∞, p_∞/p_w, and T_∞/T_w for a solution to exist changes abruptly when the flow passes the sonic speed is studied analytically. When the uniform flow blowing against the condensed phase from infinity is subsonic, it is shown that a possible flow is of Knudsen-layer type, that is, the length scale of variation of the variables is of the order of the mean free path and the variation is only appreciable in the neighbourhood of the condensed phase of the order of the mean free path. It is confirmed numerically that a Knudsen-layer-type solution exists also for the supersonic flow. Under the assumption that a Knudsen-layer-type solution exists when the set of the parameters lies on a surface in the three dimensional parameter space, the general solution in a transonic region on the supersonic side is constructed analytically. The solution consists of a Knudsen-layer-type solution and a slowly varying solution, *i.e.*, a part of a weak shock wave or a supersonic accelerating flow. There is a freedom for the choice of a semi-infinite part of a shock wave or accelerating flow as the slowly varying solution. In other words, a family of solutions with various slowly varying solutions can be constructed for a given Knudsen-layer-type solution. This explains the abrupt change at the sonic point of the parameter range, from a surface to a domain, of existence of a solution. Thus we understand the complete feature of the problem around the B-curve (except in a thin wedge region on $M_\infty = 1$ [see the paragraph after Eq. (3.52)]). A similar abrupt change of the parameter range of the existence of a solution is found when transition from evaporation to condensation is considered (Sone, 1978); an evaporating flow exists on a curve, but a condensing flow on a surface in the parameter space M_∞, p_∞/p_w, and T_∞/T_w. This change is also due to the existence of a slowly varying solution, but of different kind, in a slow condensing flow.

In the present paper, we assumed the existence of a Knudsen-layer-type solution, which is not proved yet mathematically, though numerically studied in detail. (The computation carried out in the present study requires one year or more by a fast workstation such as HP9000/C180 and DEC Alpha Station 500/500.) The assumption is the nonlinear version of the existence and uniqueness theorem for the half-space problem of the linearized Boltzmann equation studied by Bardos *et al.* (1986), Cercignani (1986), Coron *et al.* (1988), and Golse and Poupaud (1989). This is a similar situation to that where a generalized slip flow theory (Sone, 1969, 1971) or the transition from evaporation to condensation (Sone, 1978) was studied; the existence and uniqueness theorem was not available then.

Acknowledgements

The main part of this work was done while the senior author (Y. S.) was staying at Université Paris VII as an invited professor.

He expresses his hearty thanks to Professor Claude Bardos for his interest in this work and kind hospitality during the stay.

REFERENCES

AOKI K., DOI T., 1994, High-speed vapor flows condensing on a plane condensed phase in the presence of a noncondensable gas, *Rarefied Gas Dynamics: Theory and Simulations / Progress in Astronautics and Aeronautics*, **159**, Shizgal B. D., Weaver D. P., Eds., AIAA, Washington, DC, 521–536.

AOKI K., NISHINO K., SONE Y., SUGIMOTO H., 1991, Numerical analysis of steady flows of a gas condensing on or evaporating from its plane condensed phase on the basis of kinetic theory: Effect of gas motion along the condensed phase, *Phys. Fluids A*, **3**, 2260–2275.

AOKI K., SONE Y., 1991, Gas flows around the condensed phase with strong evaporation or condensation –Fluid dynamic equation and its boundary condition on the interface and their application–, *Advances in Kinetic Theory and Continuum Mechanics*, Gatignol R., Soubbaramayer, Eds., Springer-Verlag, Berlin, 43–54.

AOKI K., SONE Y., YAMADA T., 1990, Numerical analysis of gas flows condensing on its plane condensed phase on the basis of kinetic theory, *Phys. Fluids A*, **2**, 1867–1878.

BARDOS C., CAFLIŠH R. E., NICOLAENKO B., 1986, The Milne and Kramers problems for the Boltzmann equation of a hard sphere gas, *Commun. Pure Appl. Math.*, **39**, 323-352.

BHATNAGAR P. L., GROSS E. P., KROOK M., 1954, A model for collision processes in gases. I. Small amplitude processes in charged and neutral one-component systems, *Phys. Rev.*, **94**, 511–525.

CAFLIŠH R. E., NICOLAENKO B., 1982, Shock profiles solutions of the Boltzmann equation, *Commun. Math. Phys.*, **86**, 161–194.

CERCIGNANI C., 1970, Bifurcation problems in fluid-mechanics, *Meccanica*, **5**, 7–16.

CERCIGNANI C., 1986, Half-space problem in the kinetic theory of gases, *Trends in Applications of Pure Mathematics to Mechanics*, Kröner E., Kirchgässner K., Eds., Springer-Verlag, Berlin, 35–50.

CERCIGNANI C., 1988, *The Boltzmann Equation and Its Applications*, Springer-Verlag, Berlin.

CORON F., GOLSE F., SULEM C., 1988, A classification of well-posed kinetic layer problems, *Commun. Pure Appl. Math.*, **41**, 409–435.

GHAOUI Z., GOLSE F., 1997, Une classification des couches limites cinétique par le théorème de l'indice, *CR Acad. Sci. Paris, série I*, **324**, 343–348.

GOLSE F., POUPAUD F., 1989, Stationary solutions of the linearized Boltzmann equation in a half-space, *Math. Methods Appl. Sci.*, **11**, 483-502.

GRAD H., 1958, Principles of the kinetic theory of gases, *Handbuch der Physik*, Flügge S., Ed., Springer-Verlag, Berlin, **12**, 205-294.

GRAD H., 1969, Singular and nonuniform limits of solutions of the Boltzmann equation, *Transport Theory*, Bellman R., Birkhoff G., Abu-Shumays I., Eds., American Mathematical Society, Providence, RI, 269-308.

KOGAN M. N., 1958, On the equations of motion of a rarefied gas, *Appl. Math. Mech.*, **22**, 597–607.

KOGAN M. N., ABRAMOV A. A., 1991, Direct simulation solution of the strong evaporation and condensation problem, *Rarefied Gas Dynamics*, Beylich A. E., Ed., VCH Verlagsgesellschaft, Weinheim, 1251–1257.

KRYUKOV A. P., 1991, Strong subsonic and supersonic condensation on a plane surface, *Rarefied Gas Dynamics*, Beylich A. E., Ed., VCH Verlagsgesellschaft, Weinheim, 1278–1284.

MIYAKE Y., 1980, Master Thesis, Department of Aeronautical Engineering, Kyoto University (in Japanese).

OHWADA T., 1993, Structure of normal shock waves: Direct numerical analysis of the Boltzmann equation for hard-sphere molecules, *Phys. Fluids A*, **5**, 217–234.

OHWADA T., SONE Y., 1992, Analysis of thermal stress slip flow and negative thermophoresis using the Boltzmann equation for hard-sphere molecules, *Eur. J. Mech., B/Fluids*, **11**, 389-414.

PEKERIS C. L., ALTERMAN Z., 1957, Solution of the Boltzmann-Hilbert integral equation II. The coefficients of viscosity and heat conduction, *Proceedings of the National Academy of Sciences of the U. S. A.*, **43**, 998-1007.

SONE Y., 1969, Asymptotic theory of flow of rarefied gas over a smooth boundary I, *Rarefied Gas Dynamics*, Trilling L., Wachman H. Y., Eds., Academic, New York, 243-253.

SONE Y., 1971, Asymptotic theory of flow of rarefied gas over a smooth boundary II, *Rarefied Gas Dynamics*, Dini D., Ed., Editrice Tecnico Scientifica, Pisa, 737–749.

SONE Y., 1978, Kinetic theory of evaporation and condensation —Linear and nonlinear problems—, *J. Phys. Soc. Jpn*, **45** , 315–320.

SONE Y., AOKI K., 1994, *Molecular Gas Dynamics*, Asakura, Tokyo (in Japanese).

SONE Y., AOKI K., DOI T., 1992, Kinetic theory analysis of gas flows condensing on a plane condensed phase: Case of a mixture of a vapor and a noncondensable gas, *Transp. Theory Stat. Phys.*, **21**, 297–328.

SONE Y., AOKI K., TAKATA S., SUGIMOTO H., BOBYLEV A. V., 1996, Inappropriateness of the heat-conduction equation for description of a temperature field of a stationary gas in the continuum limit: Examination by asymptotic analysis and numerical computation of the Boltzmann equation, *Phys. Fluids*, **8**, 628–638.

SONE Y., AOKI K., YAMASHITA I., 1986, A study of unsteady strong condensation on a plane condensed phase with special interest in formation of steady profile, *Rarefied Gas Dynamics*, Boffi V., Cercignani C., Eds., Teubner, Stuttgart, **II**, 323–333.

WELANDER P., 1954, On the temperature jump in a rarefied gas, *Ark. Fys.*, **7**, 507–553.

(Manuscript received November 26, 1997;
revised February 24, 1998;
accepted February 26, 1998.)